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REFRACTION AND THE DETERMINATION OF
SECOND ORDER ASTRONOMIC LATITUDES

REFRACTION AND THE DETERMINATION OF SECOND ORDER ASTRONOMIC LATITUDES

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Dedicated

to

C. B. DeBiscay

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SUMMARY

This study investigated the effect of using different refraction models upon the second order astronomic latitudes obtained from observed stars. The three refraction models used were those of Baldini, Garfinkel and the United States Coast and Geodetic Survey. It was found when using the method of observing a north and a south star to obtain the final latitude, that no significant difference existed in the final combined values of latitude using any of the refraction models. However, the difference between the north star and south star values of latitude were slightly larger using the Garfinkel Model than when the other two models were used.

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CHAPTER 1

PURPOSE AND SCOPE

The major investigation undertaken in this thesis is the examination of the effect of the refraction values given by three selected refraction models upon the astronomical latitude determined from a set of star observations at a given station. The three refraction models which were used are those of the United States Coast and Geodetic Survey, [3] Baldini [1] and Garfinkel [5]. The primary objective of the project has been to determine whether any significant difference in the final value of the astronomic latitude exists when each of the refraction models is used. This would also include any effect on the standard error of the mean latitude for each station. The next objective of the study was to determine if any model was sufficiently accurate to enable the observer to obtain his latitude position by observing one single star, either north or south, in place of the current method of observing both a north and south star. Finally, it was attempted to suggest a method which would include an adjustment of the refraction coefficients from the observed data at a station. The Garfinkel model was the model used for this adjustment project.

The observed data for thirteen second order latitude stations in Southwestern Ohio were used for this investigation. Each of these stations was reduced using each refraction model and the results compared. This is listed in table 4. The IBM 7094 electronic computer was used for these reductions and the programs were written in the SCATRAN language.

In addition to the actual station reductions various aspects of the refraction models were investigated to determine the effects caused

by the variation of certain assumed values or constants upon the computed refraction. This was performed on the Baldini model to determine the effects of the relative humidity and the wave length of the observed light and on the Garfinkel model for certain constants. The refraction values computed by each of the three models for various zenith distances were compared for a range of temperature and pressure combinations for selected conditions.

This report will be generally divided into five main sections in addition to this one. The second section will contain some basic definitions and describe selected classical theories of the variation of the earth's atmosphere. The third section will briefly describe the basic assumptions that are made for the three refraction models and list their final computation equations. This section will also include the investigations performed on the various models and the results of the comparison of their refraction values. The fourth section will deal with the latitude reduction methods used in this study and the results obtained. The fifth major section will be devoted to the adjustment procedures and results obtained using the station data and a form of the Garfinkel Refraction Equation. The final section will serve as a summary and a statement of the major conclusions of the investigation.

CHAPTER 2

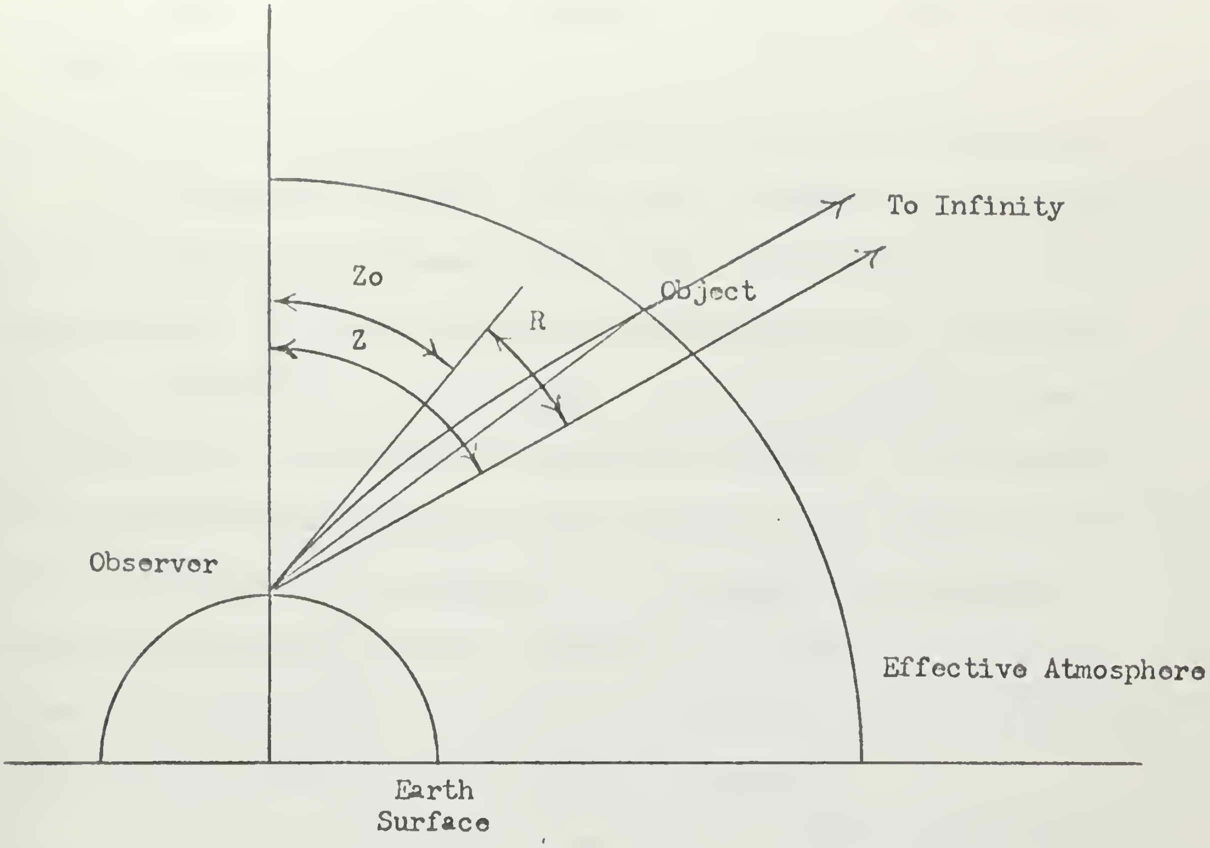
BACKGROUND

2.1 General

For this paper, astronomic refraction will be defined as the apparent displacement of an object that results from light rays from a source outside the atmosphere being bent in passing through the atmosphere.[6] The measure of this quantity is the change produced in the direction of the light rays. The total amount of this change depends on the density of the various strata of air through which the light rays pass and upon the direction of these strata with respect to the vertical. (2,7)

It is the consideration of the constitution of the atmosphere which gives rise to the differences in the various proposed refraction models. The atmosphere of the earth is known to be more dense near the surface of the earth and gradually decreases in density to its upper limit. For our purposes the upper limit of the atmosphere will be the point at which the density is so low that its effect on a ray of light is negligible. Since this condition causes the ray of light to always pass from a medium of lower density to one of higher density, the result will be that all observed objects will appear to be higher above the horizon than they actually are. Figure 1 shows the path of a ray of light from the time it enters the atmosphere until it is seen by an observer. (2,6,7)

The index of refraction for a given media of constant density shall be defined here as the ratio of the sine of the angle of incidence to the sine of the angle of refraction, when these angles are measured with respect to a normal to the boundary surface where the light ray



R = Refraction

Z_o = Observed Zenith Distance

Z = True Zenith Distance

Figure 1. Refraction Caused by the Earth's Atmosphere

passes into the given media from a vacuum. The angle of incidence is measured at the boundary surface before the ray passes into the media and the angle of refraction is measured after it has passed through the boundary surface.

2.2 Classical Theories of the Variation of the Earth's Atmosphere.

The density variation of the earth's atmosphere from the surface to the upper limit is assumed to be subject to the conditions of equilibrium. Therefore, it can be generally stated that the density varies with temperature and pressure. Since the astronomic refraction is dependent on the density, it may also be related to the variation of the temperature and pressure of the atmosphere. The various refraction tables which have been constructed for astronomical use rest upon different expressions for the variation of the density of the atmosphere with the altitude. Four of the classical theories for this variation will be reviewed in the following paragraphs.

The first real attempt to describe the variation of the atmosphere was put forth by Newton. This hypothesis assumed a constant temperature at all altitudes. The relation has been discarded in modern times and therefore this theory should be considered only in an historical sense.[7]

Bessel proposed that the density could be expressed in terms of altitude only and therefore assumed no law of temperature. The ratio of the densities at two altitudes is expressed as

$$\frac{\rho}{\rho_1} = e^{-K\frac{h}{h_1}}$$

where ρ is the density at altitude h and the subscript one refers to the conditions of the base station. K is a constant factor used for computations (K=0.9649).[7]

Ivory put forth the hypothesis that the temperature diminishes at a uniform rate with the height at all altitudes. This represents the ratio of the densities as

$$\frac{\rho}{\rho_1} = \left(1 - \frac{h}{h_0}\right)^y$$

where

$$T = T_1 (1 - \beta h) \quad \text{and} \quad y = \frac{1}{\beta h_1} - 1 .$$

In this relation h_0 is the altitude at which the atmosphere would terinate, β is a constant factor used to relate the temperature (T) at a given altitude (h) to the temperature at the base station (T_1), and the exponent y is expressed in terms of the height of the base station (h_1) and β .

$$y + 1 = \frac{h_0}{h_1} = \frac{h_0}{\gamma T_1}$$

where the value of γ is a constant, depending on the elasticity of air at a given temperature and density. (7)

The last classical hypothesis to be given here states that the temperature diminishes by a constant fraction of its absolute amount for every unit increase in altitude. This is in the form of a constant geometrical progression which may be represented, in terms of the ratio of densities, as

$$\frac{\rho}{\rho_1} = e^{-V}$$

where

$$V = \frac{g}{\beta \gamma T_1} \left(e^{\beta h} - 1\right) - \beta h$$

The small g represents the value of gravity and all the other terms have been previously defined. (7)

These theories have been presented as a background for the discussion of the specific refraction models with which we are concerned in the next section.

CHAPTER 3

REFRACTION MODELS INVESTIGATED FOR THIS PROJECT

3.1 General

The descriptions of the three refraction models which follow are not intended to be detailed discussions of the procedures used to derive the refraction equations. Rather, they simply attempt to give the basic assumptions upon which the models are based and the final forms of the refraction equations used for computations. It will be attempted to point out any limitations introduced into the models and any similarities which may exist.

The computer programs used for the practical computations with each refraction model will be generally described after the basic discussion of each model. The statements listings and required input data for each of the programs covered may be found in Appendix I.

To save repetition with each refraction model, some basic terms common to all three models will be defined here. The quantity R will always represent the astronomic refraction. Z will be the zenith distance corrected for refraction and Zo will represent the observed zenith distance. T and P will represent the absolute temperature and pressure respectively. When these last two quantities appear with the subscript (o) they will refer to the conditions at the observation station. All other quantities will be defined when they appear in the discussion for each model.

3.2 The United States Coast and Geodetic Survey Refraction Model

This refraction model is based on a theoretical background which is combined with physical data to produce a proposed composition of the atmosphere. This model follows that proposed by Willis [11] and is

generally based upon the theories of Newcomb.[7]

First, the pressure and temperature of the atmosphere are assumed to vary with respect to the density in the relation given for a perfect gas. This relationship is given as

$$\rho = \frac{kP}{T}$$

where ρ is the density and k is a constant depending on the units used and the gas in question. Willis assumed that the temperature and pressure in free air varies according to the relation that

$$\left(\frac{T}{T_o}\right) = 0.670+0.5295\left(\frac{P}{Po}\right)-0.2625\left(\frac{P}{Po}\right)^2$$

where the relative temperature at a point is given as a function of the relative pressure. The relative pressure is known to be closely related to the mass heights of the atmosphere. The density of the atmosphere at a point could then be computed after the pressure and temperature were obtained.[3,18]

The derivations for this model are based upon the assumption that the atmosphere takes the form of concentric spherical surfaces. These may be described as horizontal strata which are of equal density and follow level surfaces. Any ellipticity of the actual physical surface is neglected.[3,11]

Up to a zenith distance of 60 degrees a first approximation of the refraction formula may be given as

$$R= \frac{kB}{T} \tan Z$$

where B is the barometric pressure at the point of observation and k is a constant whose value depends on the units of temperature and pressure. For a more accurate value of the refraction and for zenith distances over 60 degrees more terms must be added to this basic expression. The form used by the Coast and Geodetic Survey is

$$R = \frac{\alpha}{1+\alpha}(\tan Z)\left[1-(\nu-\tfrac{1}{2}\alpha)\sec^2 Z\right]$$

which extends the reliable application of the refraction formula to a zenith distance of 70 degrees or more. The quantity $\frac{\alpha}{1+\alpha}$ nearly equals the refractive index of the air. The quantity ν is based on the radius of curvature of the strata and properties of the atmosphere at the station. This is roughly equal to $\frac{1}{800}$. This formula may be expressed as a part of an infinite series which is given as

$$R = \frac{\alpha}{1+\alpha}(\tan Z)(1-C_2 \sec^2 Z + C_4 \sec^4 Z - C_6 \sec^6 Z + \text{--}).$$

The C_2 coefficient is the quantity $\nu-\frac{1}{2}\alpha$ and the remaining coefficients depent on the constitution of the atmosphere. Even after these coefficients have been determined for an assumed state of things, the series ceases to converge before the horizon is reached. It may be used practically for computations up to a zenith distance of 83 or 84 degrees.

For the practical computation of the refraction a logarithmic form of the above refraction formula is used. This may be represented as

$$\log_{10} R = \log_{10}(\tan Z) + \log_{10}\left(\frac{\alpha}{1+\alpha}\right) + \log_{10}(1-C_2\sec^2 Z) + \beta + \gamma + (\lambda-1)(\gamma+\tfrac{\beta}{10})$$

where the quantities represented by the logarithms are terms of the refraction formula given above. β and γ represent departures from the standard atmospheric conditions of pressure and temperature. The term containing C_2 is added as a correction to the $\alpha/(1+\alpha)$ term which varies slowly with moderate zenith distance. However, when this term is introduced it does not allow the whole refraction equation to be taken as proportional to temperature and pressure. This is taken into account by the $(\lambda-1)(\gamma+\frac{\beta}{10})$ term in the expression. This last term does not become important except at the larger zenith distances.[3, 11]

The tables published for this refraction model were derived for

a pressure of 760 millimeters, a temperature of 10 degrees centigrade, a wave length for yellow light of 0.578 micron and a relative humidity of 60 per cent. Correction tables are provided for differences in the observed temperature and pressure from these selected values. These tables were used in the computations for this investigation.[3,11]

The computer program used for the Coast and Geodetic Survey Refraction Model was fairly simple and straightforward. The refraction values given in Table V of the U. S. Coast and Geodetic Survey Publication No. 237[3] were read into the main program for every ten minutes of arc. Since these values were computed for a given set of conditions, correction coefficients for temperature and pressure variations had to be used. These were taken from Tables VI and VII of the above publication. These correction coefficients were introduced into the subprogram using literal statements. These values were limited to a reasonable range of temperature and pressure for the reduction of the observed data. The range of these values was expanded for the comparison of the three refraction models at various extremes of temperature and pressure. The subprogram was made up of three search loops which computed the correction values for the given temperature and pressure and the refraction value for the observed zenith distance. All of these factors were then multiplied together to obtain the astronomic refraction.

3.3 The Baldini Refraction Model

The refraction equations in this model are derived for an object inside the atmosphere but the final form of the equations is such that they will hold for an object at infinity. Since it is assumed that all stars are at infinity, this model may be used to compute

astronomic refraction.

For his derivations Baldini assumed a spherical earth with an atmosphere arranged in spherical layers. He adopted the conditions that the altitude of the atmosphere would always be less that 64 kilometers and the observed zenith distance would be less than 75 degrees. This altitude limit represents the maximum height of strata over which the refracting power of the atmosphere is assumed to be zero.

In this model Baldini defines n as the index of refraction of one layer in the atmosphere and n + dn as the index of refraction of the next layer. He states his basic refraction equation in the form

$$R = -\frac{1}{\sin 1''} \iint \tan Z \, \frac{dn}{n}$$

where Z is the zenith distance of the object in the first layer of the atmosphere with an index of refraction n. To solve this equation for the refraction expressions for dn/n and tan Z are required. To obtain these, since both are a function of the density of the atmosphere, Baldini used the relationship for density at various altitudes that

$$\frac{\rho}{\rho_o} = e^{-\frac{h}{h_o}}$$

in which h_o is a constant and ρ_o is the density at the observer's location on the earth's surface. ρ is the density at an altitude h in the atmosphere. After determining a value for h_o this expression becomes

$$\rho = \rho_o e^{-0.1082h}$$

where h is given in kilometers. This expression is similar in form to the relation proposed by Bessel.(1)

Using these relationships the final form of the refraction equation for an object at infinity becomes

$$R = A_o(n_o-1)\tan Z_o + A_1(n_o-1)\tan^3 Z_o + A_2(n_o-1)\tan^5 Z_o$$

with the quantities A_o, A_1, and A_2 given as

$$A_o = +0.99827$$

$$A_1 = -0.00130$$

$$A_2 = +0.000006$$

from the derivations. The factor $(n_o - 1)$, with n_o as the index of refraction at the observer's station, is computed from the equation of Barrel and Sears. This computation is done in two steps with a preliminary computation being performed for the refractive index of standard air at optical frequencies. This value, designated as n_g, is obtained using the formula

$$(n_g - 1)\, 10^7 = 2.87604 + \frac{16.288}{\lambda^2} + \frac{0.136}{\lambda^4}$$

where λ represents the light wave length in microns. This is then converted to the ambient conditions using the relationship that

$$(n_o - 1) = \frac{n_g - 1}{1 + \alpha t}\,\frac{P}{760} - \frac{0.000000055\, e}{1 + \alpha t}$$

where e is the partial pressure of water vapor in the air, α is the coefficient of expansion of air (0.00367) and t is the temperature in degrees centigrade. The quantity e is obtained from the relative humidity and a table of saturated vapor pressures for various temperatures.(1)

The computer program used for this refraction model is self-contained in its own subprogram. The computations using the Baldini Refraction Model were all done for an assumed relative humidity of 60 per cent and a light wave length of 0.578 micron. A table of saturated vapor pressures for selected temperatures was entered in the subprogram through the use of a literals statement. A search loop similar to those used in the Coast and Geodetic Survey Model program was employed to obtain the saturated value of e at the desired temperature. This value was modified for use in the program by multiplying it by the rela-

tive humidity. The quantity $(n_0 - 1)$ was then computed and the astronomic refraction obtained from the final refraction formula given above.

3.31 Effects of Relative Humidity and Light Wave Length on the Baldini Model

An investigation was run on this subroutine to determine the effect of differences in the relative humidity and the wave length of light used for the observation. It was found that the refraction varied very little for a range of relative humidity from 10 to 90 per cent. This was for a zenith distance range of 0 to 85 degrees with comparisons being made at every five degrees of zenith distance. These results are included in Table 1 at the end of this subsection.

The effect of the wave length of light was examined for wave lengths ranging from 0.475 to 0.693 micron which is the range of the visible spectrum. There is a larger variation due to the wave length effect than was found with the relative humidity. The differences were as large as three seconds of arc at a zenith distance of 75 degrees. The results of this investigation are included in Table 2 at the end of this subsection.

From the results of these investigations it has been assumed that the values used in the computation program were sufficiently accurate as there was very little variation in the refraction due to the relative humidity and nearly all observable stars emit light in the yellow wave length region of 0.578 micron.

TABLE 1

EFFECTS OF RELATIVE HUMIDITY
ON THE
BALDINI REFRACTION MODEL

Zenith Distance (degrees)	Refraction (seconds)								
Relative Humidity	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	5.28	5.28	5.28	5.28	5.28	5.28	5.28	5.28	5.28
10	10.25	10.24	10.24	10.24	10.24	10.24	10.23	10.23	10.23
15	15.57	15.57	15.56	15.56	15.56	15.55	15.55	15.55	15.55
20	21.15	21.14	21.14	21.13	21.13	21.13	21.12	21.12	21.12
25	27.09	27.08	27.08	27.07	27.07	27.06	27.06	27.06	27.05
30	33.53	33.53	33.52	33.52	33.51	33.50	33.50	33.49	33.49
35	40.66	40.65	40.65	40.64	40.63	40.63	40.62	40.61	40.60
40	48.71	48.70	48.70	48.69	48.68	48.67	48.66	48.65	48.64
45	58.03	58.02	58.01	58.00	57.98	57.98	57.97	57.96	57.95
50	69.12	69.11	69.10	69.08	69.07	69.06	69.05	69.04	69.03
55	82.77	82.75	82.74	82.78	82.71	82.69	82.68	82.67	82.66
60	100.25	100.24	100.22	100.20	100.19	100.17	100.15	100.13	100.12
65	123.88	123.86	123.84	123.81	123.79	123.77	123.75	123.73	123.71
70	158.13	158.10	158.08	158.05	158.02	157.99	157.97	157.94	157.91
75	213.17	213.14	213.10	213.06	213.03	212.99	212.95	212.92	212.88
80	317.78	317.73	317.67	317.62	317.56	317.51	317.45	317.40	317.34
85	619.29	619.19	619.08	618.97	618.76	618.76	618.65	618.54	618.44

Temperature = 10°Centigrade   Pressure = 760 mm Hg

Wave Length = 0.578 Microns

TABLE 2

EFFECTS OF WAVE LENGTH
ON THE
BALDINI REFRACTION MODEL

Zenith Distance (degrees)	Refraction (seconds)			
Wave Length (microns)	0.475	0.505	0.578	0.693
0	0.00	0.00	0.00	0.00
5	5.12	5.11	5.08	5.05
10	10.32	10.29	10.24	10.18
15	15.69	15.64	15.55	15.47
20	21.31	21.24	21.13	21.02
25	27.29	27.21	27.06	26.92
30	33.79	33.69	33.50	33.33
35	40.97	40.85	40.63	40.41
40	49.08	48.94	48.67	48.41
45	58.47	58.30	57.98	57.67
50	69.65	69.44	69.06	68.70
55	83.40	83.15	82.69	82.26
60	101.02	100.72	100.17	99.64
65	124.83	124.45	123.77	123.12
70	159.34	158.86	157.99	157.16
75	214.80	214.15	212.99	211.86
80	320.21	319.24	317.51	315.83
85	624.03	622.14	618.76	615.48

Temperature = 10° Centigrade   Pressure = 760 mm HG
Relative Humidity = 0.60

3.4 The Garfinkel Refraction Model

The Garfinkel Model is the only one of the three used in this investigation which was designed to give refraction values for zenith distances up to and greater than 90 degrees. There are some variations in the final form of the refraction equation for this model among the agencies which use it. Two of these variations will be given after a discussion of the original form derived by Garfinkel.

Garfinkel selects a model for the variation of the density of the atmosphere that is similar to the model suggested by Ivory. In a model of this type the temperature diminishes with a uniform rate regardless of the altitude. This type of model is called a polytropic model.(5)

With the perfect gas law assumed to hold for the earth's atmosphere, the density relationship is given in its final form as

$$\rho = \left[1 - \frac{y}{n+1}\right]^n$$

where ρ is the relative density of the atmosphere at an altitude y above the observation station. The quantity n is called the polytropic index. The relative temperature is considered to diminish at a constant rate to the point where the altitude is y_1. Above this point the relative temperature is considered to be zero. This relation may be given by

$$T' = -\frac{1}{n+1}$$

where T' is the constant rate at which the relative temperature decreases. The relative temperature is then given as

$$T = 1 - \frac{y}{n+1}$$

where the condition at the observation station would be

$$T = 1 \quad \text{and} \quad y = 0.$$

It may be seen that using this relationship for the temperature the value y_1 must satisfy the relation

$$y_1 = n + 1$$

for the relative temperature to equal zero at that altitude.[5]

From this polytropic relationship Garfinkel develops a formula in which he expresses the refraction as a power series which is a function of the observed zenith distance. This equation is of the form

$$R = T_o^{\frac{1}{2}} w(B_o + B_1 W + B_2 W^2 + B_3 W^3 + ---)$$

for zenith distances that are less than 90 degrees. The terms which make up this series will be explained in the following paragraphs.[5]

In the derivation of the final form of this refraction equation there are some constants which appear. The most important of these are called K_o, β_o and γ_o. These are affected by local conditions at the observing station as well as the choice of a value for the polytropic index. If a standard set of conditions is chosen and these values computed for them, these values can be converted to the conditions at the observation station by temperature and pressure relationships.[5]

Some factors must be defined before the above three constants can be computed. When the index of refraction is denoted by μ_o at the observation station, a quantity α_o is defined from

$$\alpha_o = (\mu_o^2 - 1)/2\mu_o \quad .$$

From this factor a value C_o is defined as

$$C_o = \alpha_o(1 + \alpha_o) \quad .$$

Finally, a factor S_o is defined by

$$S_o = \frac{H_o}{r_o}$$

where Ho is the "height of the homogenous atmosphere for an ideal gas" and r_o is the radius of the earth to the observer's station. This

height, Ho, is the height to which the atmosphere would extend above the earth if its density were the same as at the point of observation and the pressure was equal to the observed pressure.(3,5)

Using these values from the preceding paragraph, the constants for the standard set of conditions can now be defined. These values are given a double subscript to indicate that they are derived for certain stipulated conditions and at standard temperature and pressure. These constants are defined as

$$\gamma_{oo} = \left[2\ S_{oo}\right]^{-\frac{1}{2}} (n + 1)^{-\frac{1}{2}}$$

$$B_{oo} = 2\alpha_{oo}\gamma_{oo}{}^{2}\ (n + 1)^{-\frac{1}{2}}$$

$$K_{oo} = 2\ C_{oo}\ \gamma_{oo}\ n(n + 1)^{-\frac{1}{2}}$$

using a chosen value for n. The variation of all the factors defined for the standard conditions, except temperature and pressure, is so small that it makes no significant changes in the values of these constants from station to station. Therefore, working values for these constants may be obtained at an observing station from

$$\gamma_{o} = \gamma_{oo}\ T^{-\frac{1}{2}}$$

$$B_{o} = B_{oo}\ W$$

$$K = K_{oo} T_{o}^{-\frac{1}{2}} W$$

where W is the "weather factor" and is obtained from

$$W = {}^{Po}/_{To^2}$$

where P_o and To represent the ratios of the pressure and temperature at the observing station to those standard conditions for which the values were originally computed. Garfinkel computed these constants using a value of 5.000 for n and obtained

$$\gamma_{oo} = 8.1578$$

$$\beta_{oo} = 0.03916$$

$$K_{oo} = 4952''$$

at a temperature of 0 degrees centigrade and a pressure of 760 millimeters.(5)

From these constants the B factors of the power series are defined as

$$B_i = K_{oo} \gamma_{oo}^{i} \gamma_i$$

where γ_i is a function of an angle theta. This angle is derived from the observed zenith distance by the relationship that

$$\cot\theta = \gamma T_o^{-\frac{1}{2}} \cot Z_o.$$

$_i$ may also be expressed in terms of a power series of $\tan\theta/2$.(5)

The refraction equation may be written for zenith distances greater than 90 degrees using the above quantities. From these factors two new values are defined. They are

$$C_1 = 2\, B_i \cos^2 \theta/_2 \cot^{2} {}^{+1} \theta/_2$$

and

$$V = \frac{P_o}{To^2} \tan^2 \theta/2$$

The refraction for a zenith distance greater than 90 degrees is then obtained from the formula

$$R = T_o^{\frac{1}{2}} V(C_o + C_1 V + C_2 V^2 + C_3 V^3 + ---)$$

which is also expressed in the form of a power series.(5)

The program which was used for most of the refraction computations was that of the Garfinkel Model as programmed by Aberdeen. This program was translated from the FORTRAN II version into the SCATRAN Language. It basically follows the summations given by Garfinkel in his article to obtain the B_i factors. The refraction is then computed using Garfinkel's power series formula as given above. This program uses updated factors for Garfinkel's basic constants. These are

$$n = 4.256$$

$$K_{oo} = 4472.8$$

$$\beta_{oo} = 0.044385$$

$$\gamma_{oo} = 8.7137$$

which are computed for the same standard conditions of temperature and pressure as those of the Garfinkel article. In a comparison of the program using the two sets of constants, it was found that this newer set of values gives a smaller amount of refraction for each selected zenith distance than when the original Garfinkel set was used. This difference has a magnitude of about 1.5 seconds at a zenith distance of 75 degrees. This comparison is contained in Appendix I. The Aberdeen constants were used for all reductions and model comparisons in this project.

Another form of the Garfinkel Refraction Model is used by the Coast and Geodetic Survey.(4) This method computes the refraction from the formula

$$R = To^{\frac{1}{2}}W\,(N_1 \tan B/2 + N_2 \tan^3 B/2 + N_3 \tan^5 B/2 + N_4 \tan^7 B/2)$$

where the constants are given the values

$$N_1 = 1050.61030$$

$$N_2 = 706.11502$$

$$N_3 = 262.06086$$

$$N_4 = 142.67293$$

and the angle B is the same as the angle θ in the Garfinkel article formulas. The γ factor used in this method is the same as that of the Aberdeen program.

This refraction method was also programmed and used in some reductions. It gives slightly lower refraction values for zenith

distances up to 70 degrees and slightly higher values for the zenith distances over this value when compared to the Aberdeen program. The difference amounted to about 0.02 seconds of arc at a zenith distance of 75 degrees. The advantage of this program over that of Aberdeen is that it is much shorter and uses about one-fourth of the computer time for the same purposes. Appendix I contains a comparison of the results of this subroutine with the Aberdeen values for one selected set of temperature and pressure conditions.

Another version of the Garfinkel refraction equation is that used by the Aeronautical Chart and Information Center (ACIC).[4] This obtains the refraction from the equation

$$R = N_1 \tan\Theta + N_2 \tan^3\Theta + N_3 \tan^5\Theta$$

in which the angle θ is the same as given in the Garfinkel article. This system uses a γ value of 8.1578 which is the same as Garfinkel's original value. This form was programmed by using the values given in Garfinkel's tables for the various theta angles as the N factors in the refraction equation. This version was used only as a comparison for the other programs given above. It is included in the comparison given in Appendix I.

This form of the refraction equation was used as a basis for the adjustment program discussed in section 5. N factors were assigned for the purposes of the adjustment and a γ factor of 8.7137 was used in that program.

3.41 Comments on the Garfinkel Program

The Garfinkel model was found to give the greatest difference between the computed values of latitude, from the observed north and south stars, of the three refraction models used. The Aberdeen sub-

routine was then used with different values of γ_{oo} but holding all other factors constant. This would be equivalent to changing the coefficients of the K_{oo} and β_{oo} terms which are computed from γ_{oo}. These coefficients are given as 2.0 by Garfinkel. It was found that a higher γ under these conditions gave less difference between the north and south star computed latitude values for all stations examined. There were different values of the factor which gave the minimum spread for each station but they all were higher than the selected value. These values ranged from 8.78 to 8.88 for the minimum closure on each station. However, the values of the factor over the 8.78 level had the effect of reversing the error in some stations when it was used to reduce all the observed stations. Therefore, any values higher than this one would be of questionable merit. This effect was to basically switch the positions of the north and south latitudes when compared to the combined value. This method of forcing less difference between the north and south stars is not given as a proof of any weakness in the model but is considered worthy of note since there is an indication that an adjustment in the constant factors for the K_{oo} and β_{oo} values by a slight amount might result in a more accurate refraction model. This would be associated with a new higher value for the gamma constant that would be somewhere between the Aberdeen value and the 8.78 quantity mentioned above. Much more study and research would be required before it could be decided whether the noted changes were either required or practicable. The idea is put forth as a possible area for future research but should not be accepted as anything more meaningful than that.

3.5 Comparison of the Refraction Values Obtained from the Three Refraction Models.

A comparison of the refraction values for the Baldini Model,

the Garfinkel Model and the Coast and Geodetic Survey Model was made for various conditions and extremes of temperature and pressure. These comparisons were made for a standard relative humidity of 60 per cent, light wave length of 0.578 micron and over a range of zenith distances from 0 to 90 degrees. It should be noted that the form of the Baldini model goes to infinity at 90 degrees and so no comparison can be made there.

It should also be noted in these comparisons that the Baldini Model was designed to be used only for observed zenith distances up to 75 degrees and that the series used for the Coast and Geodetic Survey Model becomes questionable after a zenith distance of 83 or 84 degrees is reached.[1, 3] The comparisons were made though to see how close the models did come. Table 3 lists the refraction values for various combinations of temperature and pressure and also the difference in the obtained values between the models. This is done using the Coast and Geodetic Survey Model as an arbitrary standard with which to compare the other two models.

The comparisons show that the Baldini and Coast and Geodetic Survey Models tend to agree more closely throughout the range of temperature and pressure than the Garfinkel and the Coast and Geodetic Survey Models. However, at higher temperatures the refraction values given by the Baldini Model tend to become lower than those given by the Coast and Geodetic Survey Model. At lower temperatures these values are higher than those given by the Coast and Geodetic Survey Model. The widest variation between the Baldini and Garfinkel Models occur at the higher temperatures. The relation between the Garfinkel Model and the Coast and Geodetic Survey Model at the higher zenith distances,

over 85 degrees, varies quite sharply with the temperature. In this area the Garfinkel Model gives higher values of refraction at the lower temperatures than the Coast and Geodetic Survey Model and lower ones at the higher temperatures. It was found generally that the refraction differences appear to be more effected by the variation of temperature than those of pressure. All the models give fairly close values for the refraction up to a zenith distance of about 50 degrees in which area the difference is about 0.3 seconds.

For the comparisons in Table 3 the values of the refraction were rounded to the nearest 0.1 second. This is done since the tables for the Coast and Geodetic Survey Model are only given to this accuracy. Since the errors in this range are within the acceptable limits for second order work this factor does not tend to affect the results of this investigation. The Aberdeen subroutine was used for the Garfinkel Refraction Model in this comparison.

TABLE 3

REFRACTION COMPARISONS
FOR THE
THREE REFRACTION MODELS

Zenith Distance (degrees)	Coast and Geodetic Survey	Baldini	Garfinkel	Difference from Coast and Geodetic Survey Model Baldini	Garfinkel
Temperature -5°C Pressure 711 mm Refraction in seconds					
0	0.0	0.0	0.0	0.0	0.0
5	5.0	5.0	5.0	0.0	0.0
10	10.1	10.1	10.1	0.0	0.0
15	15.3	15.4	15.4	0.1	0.1
20	20.9	20.9	20.9	0.0	0.0
25	26.7	26.8	26.8	0.1	0.1
30	33.0	33.1	33.2	0.1	0.2
35	40.0	40.1	40.2	0.1	0.2
40	48.0	48.1	48.2	0.1	0.2
45	57.2	57.3	57.4	0.1	0.2
50	68.1	68.3	68.4	0.2	0.3
55	81.5	81.8	81.9	0.3	0.4
60	98.9	99.0	99.2	0.2	0.4
65	122.1	122.4	122.6	0.3	0.5
70	156.0	156.2	156.6	0.2	0.6
75	210.3	210.6	211.3	0.3	1.0
80	313.6	313.9	315.4	0.3	1.8
85	581.6	611.7	585.8	30.1	4.2
90	2035.6	∞	2051.2	∞	15.6
Temperature -5°C Pressure 737 mm Refraction in seconds					
0	0.0	0.0	0.0	0.0	0.0
5	5.2	5.2	5.2	0.0	0.0
10	10.4	10.5	10.5	0.1	0.1
15	15.9	15.9	15.9	0.0	0.1
20	21.6	21.7	21.7	0.1	0.1
25	27.7	27.7	27.8	0.0	0.1
30	34.2	34.3	34.4	0.1	0.2
35	41.5	41.6	41.7	0.1	0.2
40	49.8	49.9	49.9	0.1	0.1
45	59.3	59.4	59.5	0.1	0.2
50	70.6	70.8	70.9	0.2	0.3
55	84.5	84.7	84.9	0.2	0.4
60	102.4	102.7	102.8	0.3	0.4
65	126.5	126.8	127.1	0.3	0.6
70	161.6	161.9	162.4	0.3	0.8
75	218.0	218.3	219.1	0.3	1.1
80	325.0	325.4	326.9	0.4	1.9
85	602.7	634.1	607.5	30.1	4.8
90	2109.6	∞	2132.7	∞	23.1

TABLE 3 (CON'T)

REFRACTION COMPARISONS
FOR THE
THREE REFRACTION MODELS
(Continued)

Zenith Distance (degrees)	Coast and Geodetic Survey	Baldini	Garfinkel	Difference from Coast and Geodetic Survey Model Baldini	Difference from Coast and Geodetic Survey Model Garfinkel
Temperature	-5°C	Pressure	760 mm	Refraction in seconds	
0	0.0	0.0	0.0	0.0	0.0
5	5.4	5.4	5.4	0.0	0.0
10	10.8	10.8	10.8	0.0	0.0
15	16.4	16.4	16.5	0.0	0.1
20	22.3	22.3	22.4	0.0	0.1
25	28.5	28.6	28.6	0.1	0.1
30	35.3	35.4	35.5	0.1	0.2
35	42.8	42.9	43.0	0.1	0.2
40	51.3	51.4	51.5	0.1	0.2
45	61.1	61.3	61.4	0.2	0.3
50	72.8	73.0	73.1	0.2	0.3
55	87.1	87.4	87.5	0.3	0.4
60	105.6	105.9	106.1	0.3	0.5
65	130.4	130.8	131.1	0.4	0.7
70	166.6	167.0	167.4	0.4	0.8
75	224.7	225.1	225.9	0.4	1.2
80	335.1	335.5	337.2	0.4	2.1
85	621.4	653.9	626.7	32.5	5.3
90	2174.8	∞	2205.3	∞	30.5
Temperature	-5°C	Pressure	787 mm	Refraction in seconds	
0	0.0	0.0	0.0	0.0	0.0
5	5.6	5.6	5.6	0.0	0.0
10	11.2	11.2	11.2	0.0	0.0
15	17.0	17.0	17.0	0.0	0.0
20	23.1	23.1	23.1	0.0	0.0
25	29.5	29.6	29.7	0.1	0.2
30	36.5	36.7	36.7	0.2	0.2
35	44.3	44.5	44.5	0.2	0.2
40	53.2	53.3	53.3	0.1	0.1
45	63.3	63.5	63.5	0.2	0.2
50	74.5	75.6	75.7	0.1	0.2
55	90.3	90.5	90.6	0.2	0.3
60	109.4	109.6	109.8	0.2	0.4
65	135.1	135.5	135.7	0.4	0.6
70	172.6	172.9	173.4	0.3	0.8
75	232.8	233.1	233.9	0.3	1.1
80	347.1	347.5	349.2	0.4	2.1
85	643.7	677.2	649.3	33.5	5.7
90	2253.1	∞	2291.1	∞	38.0

TABLE 3 (CON'T)

REFRACTION COMPARISONS
FOR THE
THREE REFRACTION MODELS
(Continued)

Zenith Distance (degrees)	Coast and Geodetic Survey	Baldini	Garfinkel	Difference from Coast and Geodetic Survey Model Baldini	Garfinkel
Temperature	10°C	Pressure	711 mm Refraction in seconds		
0	0.0	0.0	0.0	0.0	0.0
5	4.8	4.8	4.8	0.0	0.0
10	9.5	9.6	9.6	0.1	0.1
15	14.5	14.6	14.6	0.1	0.1
20	19.7	19.8	19.8	0.1	0.1
25	25.3	25.3	25.4	0.0	0.1
30	31.3	31.3	31.4	0.0	0.1
35	37.9	38.0	38.1	0.1	0.2
40	45.5	45.5	45.6	0.0	0.1
45	54.2	54.2	54.4	0.0	0.2
50	64.5	64.6	64.8	0.1	0.3
55	77.2	77.4	77.5	0.2	0.3
60	93.6	93.7	93.9	0.1	0.3
65	115.6	115.8	116.1	0.2	0.5
70	147.7	147.8	148.2	0.1	0.5
75	199.2	199.2	199.9	0.0	0.7
80	297.0	297.0	298.0	0.0	1.0
85	550.7	578.8	551.2	28.1	0.5
90	1927.7	∞	1874.3	∞	-53.4
Temperature	10°C	Pressure	737 mm Refraction in seconds		
0	0.0	0.0	0.0	0.0	0.0
5	4.9	4.9	4.9	0.0	0.0
10	9.9	9.9	9.9	0.0	0.0
15	15.0	15.1	15.1	0.1	0.1
20	20.5	20.5	20.5	0.0	0.0
25	26.2	26.2	26.3	0.0	0.1
30	32.4	32.4	32.6	0.0	0.2
35	39.3	39.4	39.5	0.1	0.2
40	47.1	47.2	47.3	0.1	0.2
45	56.2	56.2	56.3	0.0	0.1
50	66.8	67.0	67.1	0.2	0.1
55	80.0	80.2	80.4	0.2	0.4
60	97.0	97.1	97.4	0.1	0.4
65	119.8	120.0	120.3	0.2	0.5
70	153.1	153.2	153.7	0.1	0.5
75	206.4	206.5	207.3	0.1	0.9
80	307.8	307.9	309.0	0.1	1.2
85	570.7	600.0	571.6	29.3	0.9
90	1997.7	∞	1948.1	∞	-49.6

TABLE 3 (CON'T)

REFRACTION COMPARISONS
FOR THE
THREE REFRACTION MODELS
(Continued)

Zenith Distance (degrees)	Coast and Geodetic Survey	Baldini	Garfinkel	Difference from Coast and Geodetic Survey Model Baldini	Difference from Coast and Geodetic Survey Model Garfinkel
Temperature 10°C	Pressure 760 mm	Refraction in seconds			
0	0.0	0.0	0.0	0.0	0.0
5	5.1	5.1	5.1	0.0	0.0
10	10.2	10.2	10.2	0.0	0.0
15	15.5	15.6	15.6	0.1	0.1
20	21.1	21.1	21.1	0.0	0.0
25	27.0	27.1	27.1	0.1	0.1
30	33.4	33.5	33.6	0.1	0.2
35	40.5	40.6	40.7	0.1	0.2
40	48.6	48.7	48.8	0.1	0.2
45	57.9	58.0	58.1	0.1	0.2
50	68.9	69.1	69.2	0.2	0.3
55	82.5	82.7	82.9	0.2	0.4
60	100.0	100.2	100.4	0.2	0.4
65	123.5	123.8	124.1	0.3	0.6
70	157.8	158.0	158.5	0.2	0.7
75	212.8	213.0	213.7	0.2	0.9
80	317.3	317.5	318.6	0.2	1.3
85	588.4	618.8	589.7	30.4	1.3
90	2059.5	∞	2013.7	∞	-45.8
Temperature 10°C	Pressure 787 mm	Refraction in seconds			
0	0.0	0.0	0.0	0.0	0.0
5	5.3	5.3	5.3	0.0	0.0
10	10.6	10.6	10.6	0.0	0.0
15	16.1	16.1	16.1	0.0	0.0
20	21.9	21.9	21.9	0.0	0.0
25	28.0	28.0	28.1	0.0	0.1
30	34.6	34.7	34.8	0.1	0.2
35	42.0	42.1	42.2	0.1	0.2
40	50.3	50.4	50.5	0.1	0.2
45	60.0	60.0	60.2	0.0	0.2
50	71.4	71.5	71.7	0.1	0.3
55	85.5	85.6	85.8	0.1	0.3
60	103.6	103.7	104.0	0.1	0.4
65	127.9	128.2	128.5	0.3	0.6
70	163.5	163.6	164.1	0.1	0.5
75	220.5	220.6	221.3	0.1	0.8
80	328.7	328.8	330.0	0.1	1.2
85	609.6	640.8	610.9	31.2	1.3
90	2133.6	∞	2091.1	∞	-42.5

TABLE 3 (CON'T)

REFRACTION COMPARISONS
FOR THE
THREE REFRACTION MODELS
(Continued)

Zenith Distance (degrees)	Coast and Geodetic Survey	Baldini	Garfinkel	Difference from Coast and Geodetic Survey Model Baldini	Garfinkel
Temperature	35°C	Pressure	711 mm	Refraction in seconds	
0	0.0	0.0	0.0	0.0	0.0
5	4.4	4.3	4.4	0.1	0.0
10	8.8	8.8	8.8	0.0	0.0
15	13.3	13.3	13.4	0.0	0.1
20	18.1	18.1	18.2	0.0	0.1
25	23.2	23.1	23.3	-0.1	0.1
30	28.7	28.7	28.9	0.0	0.2
35	34.8	34.8	35.0	0.2	0.2
40	41.8	41.7	41.9	-0.1	0.1
45	49.8	49.6	49.9	-0.2	0.1
50	59.3	59.1	59.5	-0.2	0.2
55	71.0	70.8	71.2	-0.2	0.2
60	86.0	85.7	86.3	-0.3	0.3
65	106.2	105.9	106.6	-0.3	0.4
70	135.7	135.2	136.1	-0.2	0.4
75	183.0	182.3	183.4	-0.7	0.4
80	272.9	271.8	272.9	-1.1	0.0
85	506.1	529.6	501.3	23.5	-4.8
90	1771.5	∞	1632.5	∞	-139.0
Temperature	35°C	Pressure	737 mm	Refraction in seconds	
0	0.0	0.0	0.0	0.0	0.0
5	4.5	4.5	4.5	0.0	0.0
10	9.1	9.1	9.1	0.0	0.0
15	13.8	13.8	13.9	0.0	0.1
20	18.8	18.7	18.9	-0.1	0.1
25	24.1	24.0	24.2	-0.1	0.1
30	29.8	29.7	29.9	-0.1	0.1
35	36.1	36.1	36.3	0.0	0.1
40	43.3	43.1	43.5	-0.2	0.2
45	51.6	51.5	51.8	-0.1	0.2
50	61.4	61.3	61.7	-0.1	0.4
55	73.5	73.4	73.8	-0.1	0.3
60	89.1	88.9	89.4	-0.2	0.3
65	110.1	109.8	110.5	-0.3	0.4
70	140.7	140.1	141.1	-0.5	0.4
75	189.7	189.0	190.1	-0.7	0.4
80	282.9	281.8	282.9	-1.1	0.0
85	524.5	549.1	519.8	24.6	-4.7
90	1835.9	∞	1695.9	∞	-140.0

TABLE 3 (CON'T)

REFRACTION COMPARISONS
FOR THE
THREE REFRACTION MODELS
(Continued)

Zenith Distance (degrees)	Coast and Geodetic Survey	Baldini	Garfinkel	Difference from Coast and Geodetic Survey Model: Baldini	Difference from Coast and Geodetic Survey Model: Garfinkel
Temperature 35°C	Pressure 760 mm	Refraction in seconds			
0	0.0	0.0	0.0	0.0	0.0
5	4.7	4.6	4.7	-0.1	0.0
10	9.4	9.4	9.4	0.0	0.0
15	14.2	14.2	14.3	0.0	0.1
20	19.4	19.3	19.5	-0.1	0.1
25	24.8	24.8	24.9	0.0	0.1
30	30.7	30.7	30.8	0.0	0.1
35	37.2	37.2	37.4	0.0	0.2
40	44.7	44.5	44.8	-0.2	0.1
45	53.2	53.1	53.4	-0.1	0.2
50	63.3	63.2	63.6	-0.1	0.3
55	75.8	75.7	76.1	-0.1	0.4
60	91.9	91.7	92.2	-0.2	0.3
65	113.5	113.3	114.0	-0.2	0.7
70	145.0	144.6	145.5	-0.4	0.5
75	195.6	194.9	196.1	-0.7	0.5
80	291.6	290.6	291.8	-1.0	0.2
85	540.7	566.3	536.2	25.6	-4.5
90	1892.7	∞	1752.2	∞	-140.5
Temperature 35°C	Pressure 787 mm	Refraction in seconds			
0	0.0	0.0	0.0	0.0	0.0
5	4.9	4.8	4.8	-0.1	-0.1
10	9.7	9.7	9.8	0.0	0.1
15	14.8	14.7	14.8	-0.1	0.0
20	20.1	20.0	20.1	-0.1	0.0
25	25.7	25.7	25.8	0.0	0.1
30	31.8	31.8	31.9	0.0	0.1
35	38.6	38.5	38.7	-0.1	0.1
40	46.3	46.1	46.4	-0.2	0.3
45	55.1	55.0	55.3	-0.1	0.3
50	65.6	65.5	65.8	-0.1	0.2
55	78.5	78.4	78.8	-0.1	0.4
60	95.2	95.0	95.5	-0.2	0.3
65	117.6	117.3	118.0	-0.3	0.4
70	150.2	149.8	150.6	-0.4	0.4
75	202.6	201.9	203.1	-0.7	0.5
80	302.1	301.0	302.2	-1.1	0.1
85	560.2	586.5	555.4	26.3	-4.8
90	1960.8	∞	1818.7	∞	-142.1

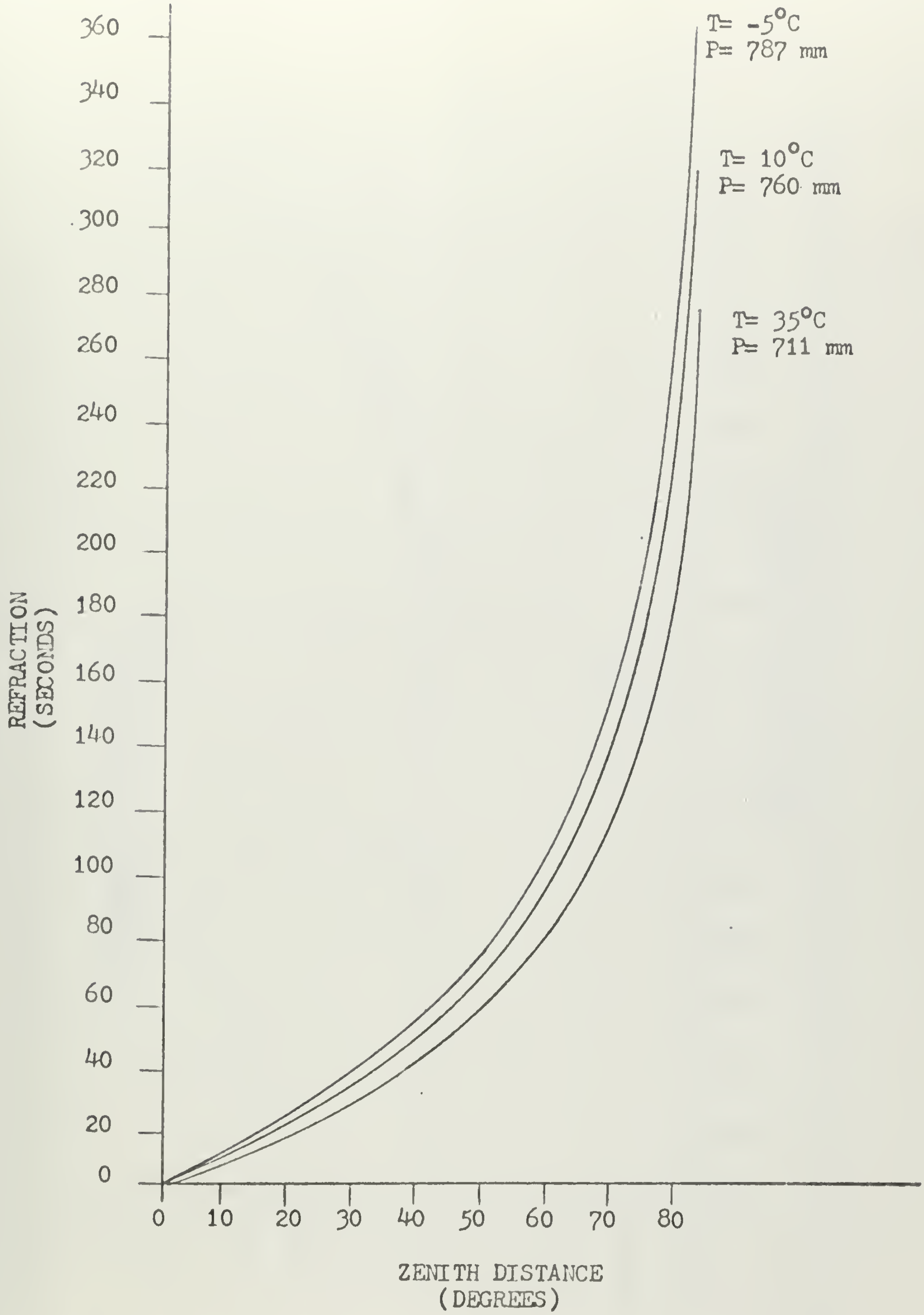


Figure 2. Refraction Variation for the Coast and Geodetic Survey Model From 0 to 80 Degrees Zenith Distance

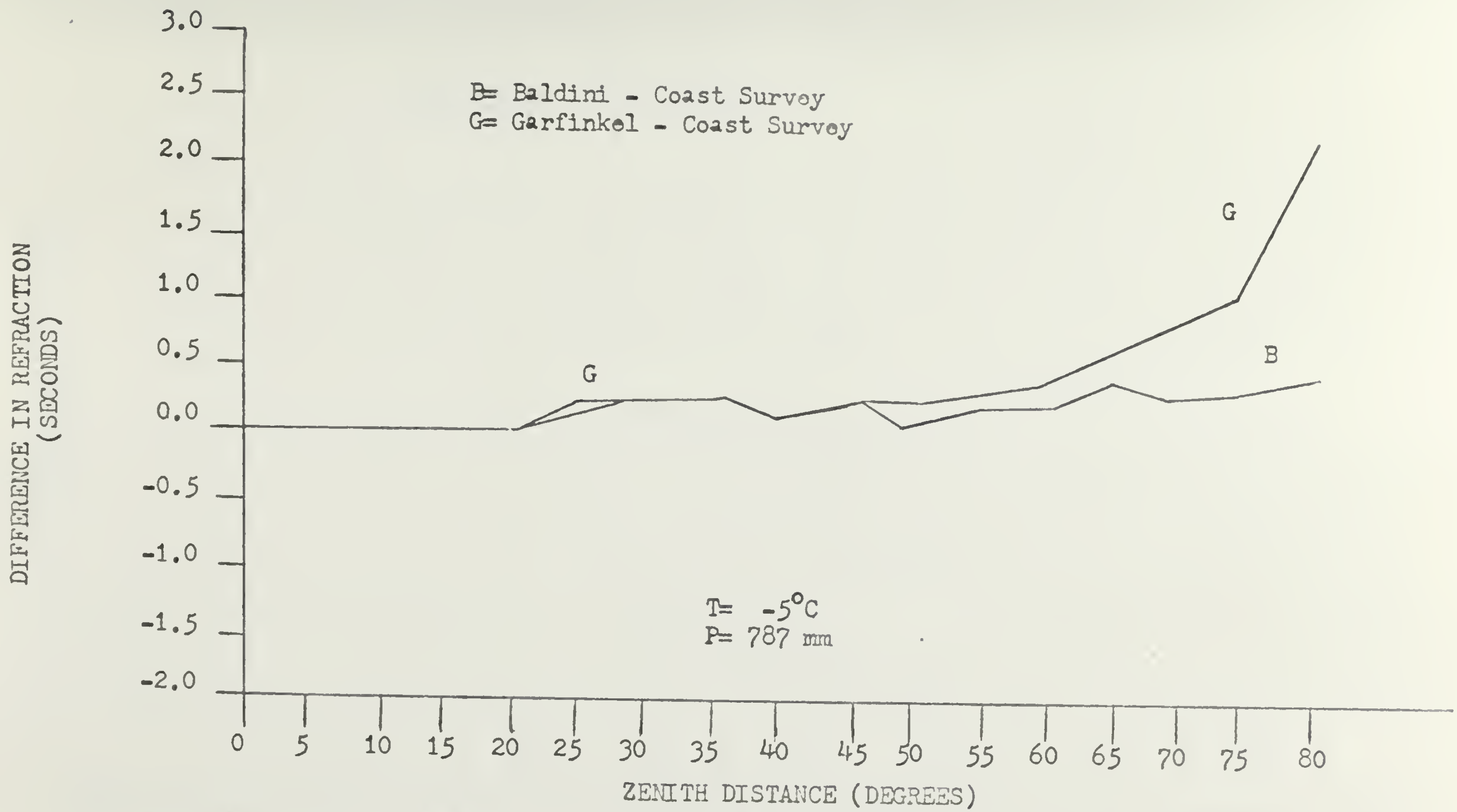


Figure 3. Refraction Differences From Coast Survey Model at Low Temperature and High Pressure

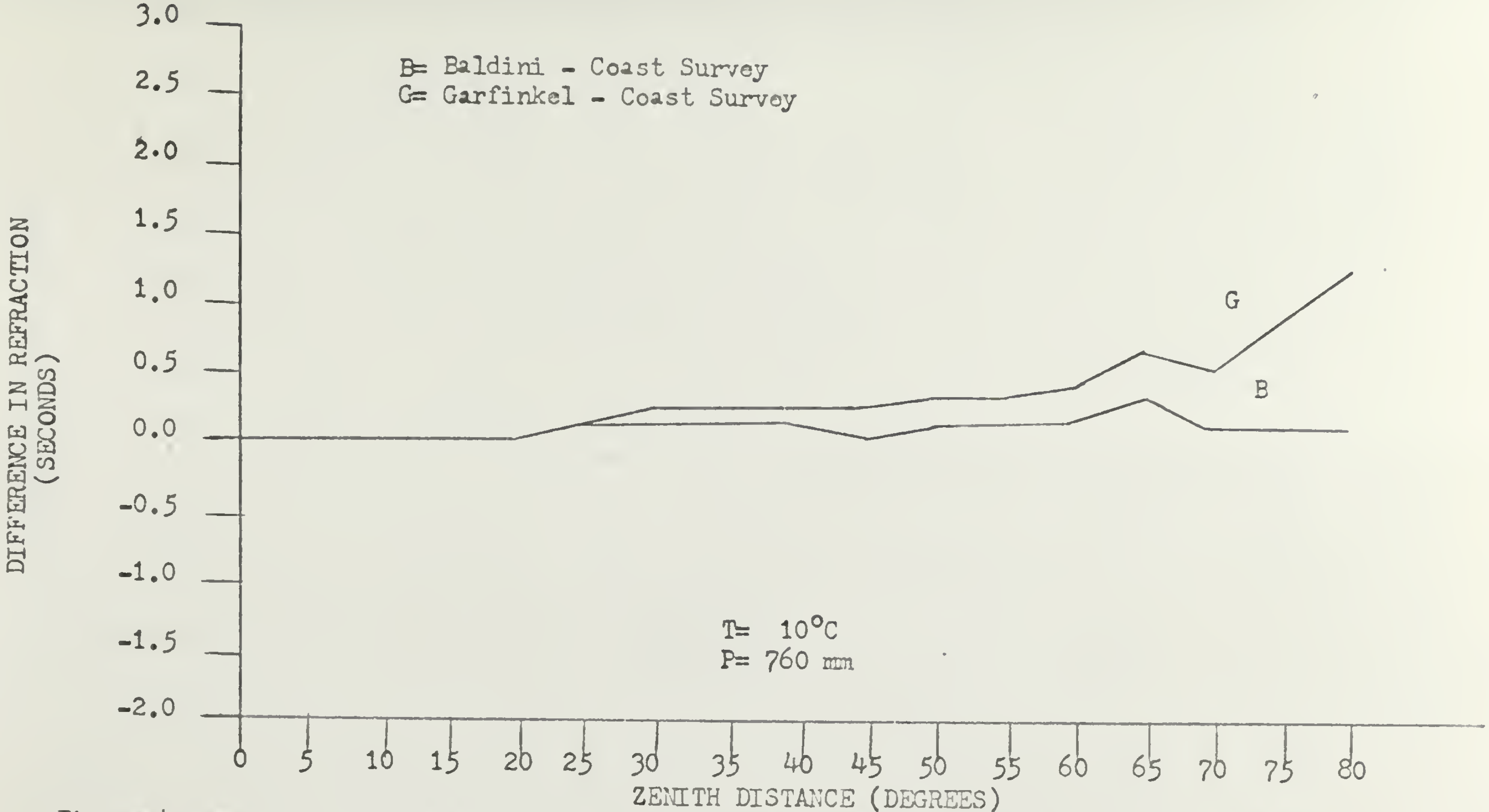


Figure 4. Refraction Differences From Coast Survey Model Near Standard Temperature and Pressure

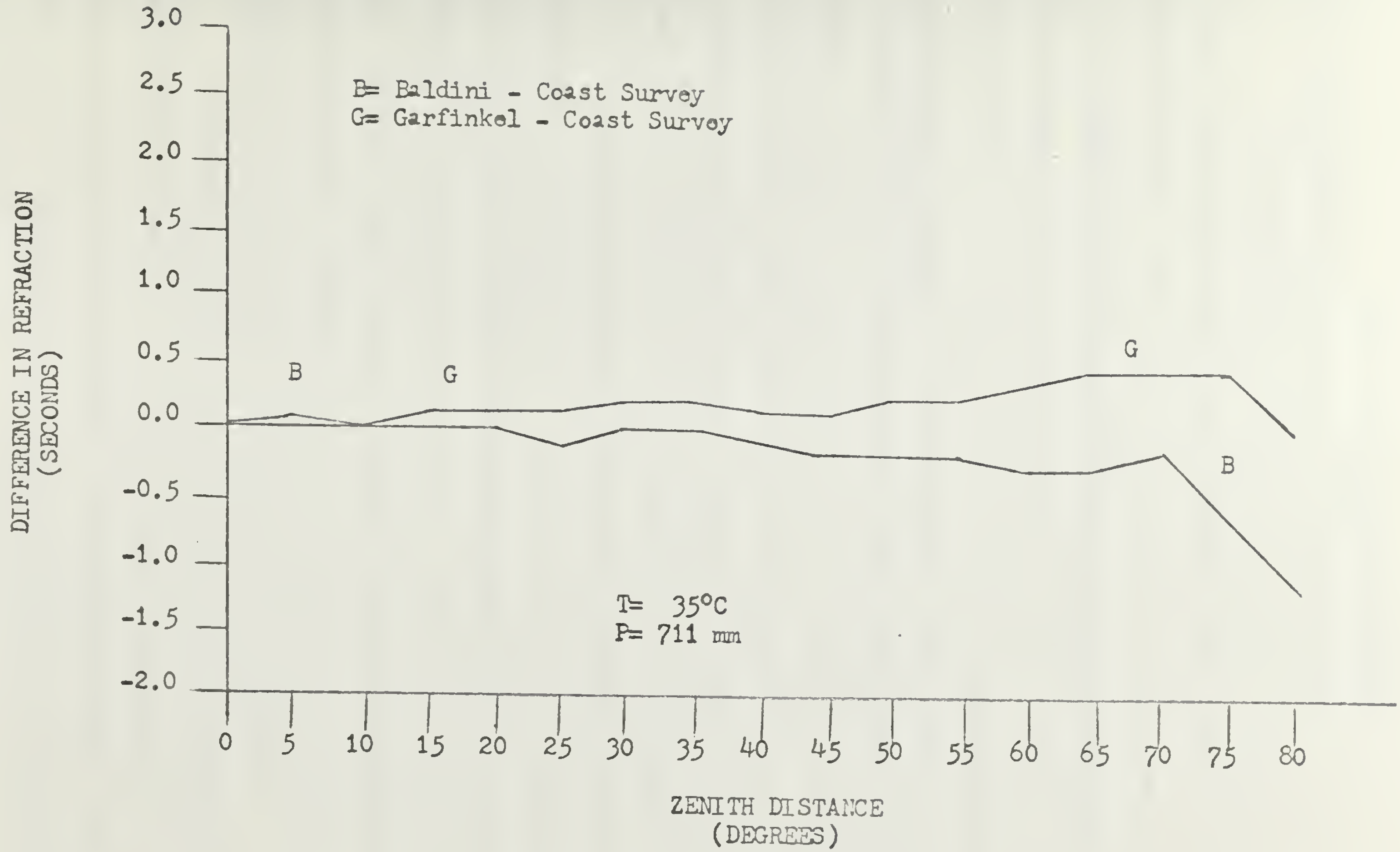


Figure 5. Refraction Differences From Coast Survey Model For High Temperature and Low Pressure

CHAPTER 4

THE LATITUDE REDUCTION METHODS

4.1 Background on the Reduction of Astronomic Data

The latitude reductions for this project were done by two computation methods which were used as a check on each other for the final values obtained. The first method employed is that given by Thorson in the Second Order Astronomical Position Determination Manual of the United States Coast and Geodetic Survey.(9) This reduction method is based in turn on a version given by Chauvenet in A Manual of Spherical and Practical Astronomy.(2) Two versions of this method were used with the only difference being the use of higher order terms in the second version to attempt to gage their effects on the final position. The second reduction method used is based strictly on the use of spherical trigonometry and the astronomical triangle. A method outlined by Chauvenet was used in the final formulation of this method.(2)

The observed data available for this investigation included observations for Polaris and a selected south star at each station. The maximum zenith distance differences between the two stars at any one station was just less than two degrees. The latitude results were reduced for both stars and then averaged to obtain the combined value for the station. A standard error of the mean value obtained in averaging each of these separate latitudes was also computed.

Astronomic latitude may be defined as the declination of the zenith of the observer. An alternate way of stating this fact is to say that the latitude is the altitude of the observer's pole. Both of these definitions of latitude will be used in explaining the reduction methods metioned above.(2)

The reduction methods used in this investigation are both based on the general spherical trigonometry formulas which are derived from the astronomical triangle. These general formulas may be stated as

$$\sin h = \sin\phi \sin\delta + \cos\phi \cos\delta \cos t$$

$$\cos t = \frac{\sin h - \sin\phi \sin\delta}{\cos\phi \cos\delta}$$

with ϕ the observer's latitude, h the altitude, δ the declination and t the hour angle of the star in question. The differences in the two methods are in certain approximations that are made and in the final method of use of the quantities. In the first reduction method, two separate means are used to reduce the north and south stars, while in the second reduction method, the same general application is made to both stars.(2,9)

4.2 Reduction Method Based on the Use of Polaris and A Circummeridian Star

This method of reduction employs different methods of reduction for the pole star and the south star. Certain approximations are made to reduce the observed position of the star to the observer's meridian and then employ the definitions given above to determine the latitude of the station.

For the pole star, the latitude will be treated as the altitude of the pole. For this reduction, the declination will be replaced by the polar distance which is defined as 90 degrees minus the declination of the star. Substituting this quantity into the first general equation given in Section 4.1 gives the expression

$$\sin h = \sin\phi \cos P + \cos\phi \sin P \cos t$$

where P represents the polar distance. As the value of P is small for Polaris (less than 2 degrees), the latitude may be developed in a series

form of ascending powers of P. This series expansion gives the option of using as many terms as are required for any desired precision. Chauvenet[2] obtains this expansion by writing the latitude equation as

$$\phi = h - x$$

where x is a small correction on the order of magnitude of P. The series expansion for x is given as

$$x = P \cos t + P^2 \sin^2 t \tan h - 1/3 \, P^3 \cos t \sin^2 t.$$

This expansion is accurate to 0.01 seconds of arc. After substitution, the final latitude expression becomes

$$\phi = h - P \cos t + \tfrac{1}{2} P^2 \sin^2 t \tan h - 1/3 \, P^3 \cos t \sin^2 t.$$

A factor may be introduced into this equation to convert all values to the same units. This would be in the last two terms where a value of sine of one second of arc, to the appropriate power, is introduced as a conversion factor if P is in seconds. Thorson does not use the last term (P^3) in his method of reduction. This limits the method to an accuracy for the final astronomic latitude of somewhere within one second of arc.[2,9]

To reduce the circummeridian star, the latitude will be obtained from the declination of the observer's zenith. Several observations of the star may be made close to the meridian and then reduced to the meridian. These altitudes observed near the meridian give nearly as accurate a result when reduced to the meridian as if the observation were taken on the meridian. This accuracy is within one second of arc for the final latitude if the observations are within 30 minutes of time from the meridian and within 0.1 seconds of arc if the observations are within 15 minutes of time from the meridian.[2]

From the first general equation of Section 4.1, Chauvenet[2]

develops an expression for the zenith distance of the observed star. Using the trigonometric relation for the hour angle that

$$\cos t = 1 - 2 \sin^2 \tfrac{1}{2} t,$$

the zenith distance may be obtained from

$$\cos Z = \sin h + \cos \phi \cos \delta (2 \sin^2 \tfrac{1}{2} t)$$

where Z is the observed zenith distance corrected for refraction. Z is introduced into the equation from the relation that on the meridian

$$Z = \phi - \delta$$

for a star south of the observer's zenith. Using the relationship that the altitude is the complement of the zenith distance, the expression may be written as

$$\cos Z = \cos Z_1 - 2 \cos \phi \cos \delta \sin^2 \tfrac{1}{2} t$$

where Z_1 is the zenith distance reduced to the meridian. This equation is then developed into a power series which in its final form is expressed as

$$Z_1 = Z - Am + Bn$$

where

$$A = \frac{\cos \phi \cos \delta}{\sin Z_1}$$

$$B = A^2 \cot Z_1$$

$$m = 2 \sin^2 \tfrac{1}{2} t$$

$$n = \sin^4 \tfrac{1}{2} t$$

in which m and n may be divided by the sine of one second to put these terms in seconds of arc. For practical computations, the value of may be used in place of Z_1 to determine the values of A and B. An assumed value of the latitude must also be used for the computations If a number of observations are taken, a series of these equations are obtained and an average value of Z_1 will result. The latitude will

then be obtained from the expression

$$\phi = \delta + Z_1$$

which may be given the alternate form

$$\phi = \delta + 90^\circ - h_1.$$

Using the values determined above, this equation reduces to

$$\phi = \delta + 90^\circ - h + Am - Bn.$$

In practice a curvature correction for the time elapsed between the direct and reverse instrument sightings is introduced. This is given as

$$m_1 = 0.0001364\ (\Delta T^s)^2$$

where ΔT^s is the elapsed time in seconds between the direct and reverse sightings. This correction is added to the m term of the above expression. Thorson does not use the last term of this expression (Bn) in his reduction. The value of the assumed latitude used in this method should be as close as possible to the expected final latitude value. (2,9)

4.3 A Method for Obtaining the Astronomic Latitudes Using Basic Spherical Trigonometry

This reduction method once again makes use of the first general equation given in section 4.1 for the solution to the astronomic triangle. The main difference between this method and the one given in the preceding section is that no series expansion is used here to obtain an expression for the latitude. Instead, two auxiliary variables, d and D, are introduced so that

$$d \sin D = \sin \delta$$

$$d \cos D = \cos \delta \ \cos t.$$

For convenience another auxiliary variable, γ, is brought into the reduction. This variable is defined by the relation

$$\phi - D = \pm \gamma .$$

After substitution the general equation may be expressed as

$$d \cos (\phi - D) = \sin h.$$

By eliminating d from the final solution, the final computation equations are obtained. These equations are given as

$$\tan D = \tan \delta \sec t$$

$$\cos \gamma = \sin h \sin D \operatorname{cosec} \delta$$

$$\phi = D \pm \gamma$$

using the auxiliary variables defined above. The first equation determines D which is taken to be always less than 90 degrees and positive or negative according to the sign of the tangent. The second equation is indeterminate as to the sign of γ since the cosine of $+\gamma$ and $-\gamma$ are the same. Therefore, two values of latitude are obtained by the third equation; however, one of these values will be over 90 degrees and can be discarded. It was found that for Polaris, the $D - \gamma$ value is to be used to obtain the latitude. For the south star, the $D + \gamma$ value gives the desired result. In these reductions a curvature correction was used for the south star to compensate for the elapsed time between the direct and reverse pointings of the instrument. This is of the form Am_1 where the quantities are the same as those introduced in the preceding section. This value was subtracted from that value for the south star latitude which resulted from the $D + \gamma$ procedure.[2,9]

4.4 The Station Reductions to Obtain the Astronomic Latitude

Each station was reduced by three methods using each refraction model. The first reduction used the procedures given by Thorson,[9] The second reduction used this same method but added the higher order

terms described in section 4.2 The third reduction employed the method outlined in section 4.3 All of these reductions were done by making use of a computer program which used the subroutines for each refraction model. The Aberdeen subroutine was used here for the Garfinkel Model. These reduction results are given for each station in Table 4. As a check on the accuracy of the reduction program, the first set of reduction results for the Coast and Geodetic Survey Model was compared to hand computed results and found to be in extremely close agreement. The slight differences, of a maximum value of 0.02 seconds of arc, may be attributed partially to the fact that the hand computations used averaged values of the observed altitudes to determine the trigonometric functions used in the reduction. In the program the function was computed directly for each observed altitude (corrected for the refraction). The agreement between the final values was regarded as being within the accuracy limits required for this investigation.

In general, the final latitude values for the north and south stars, as well as the combined latitude, agreed closely for each of the three programmed methods employed. The second and third reduction programs were generally in close agreement in their results. It was found that these two programs reduced the differences between the computed latitudes for the north and south stars by nearly 0.1 second when compared to the results of the first method. This change may be attributed to the greater accuracy of these programs.

Thorson states that if all field requirements are met, the final computations will show probable errors of less than ± 0.4 seconds of arc.[9] This would be equivelent to a standard error of about ± 0.6 seconds of arc. All of the observations used in this project were with-

in this range. The standard errors for the mean values of the latitudes determined by each reduction method are given in Table 4.

4.5 The Effects of Refraction Model Errors Upon the Computed Latitude Values.

The latitude may be computed from the observed zenith distance of a star using the definitions given in section 4.1 For the case of a star observed in the meridian, the latitude could be obtained from the equations

$$\phi = (90^\circ - \delta) + (90^\circ - Z)$$

or

$$\phi = \delta - Z$$

for a star observed north of the observer's meridian and

$$\phi = \delta + Z$$

for a star south of the observer's meridian. In these equations ϕ is the latitude, δ the declination and Z the true zenith distance. In the case of a circumpolar star observed at lower transit, the declination would have to be replaced by its supplement in the equations for the north star. The true zenith distance is obtained by adding the refraction to the observed zenith distance. This relationship is expressed by

$$Z = Z_o + R$$

where Z_o is the observed zenith distance and R is the refraction correction. Substituting this expression into the latitude equations, the relationships are given as

$$\phi = (90^\circ - \delta) + (90^\circ - Z_o - R)$$

or

$$\phi = \delta - Z_o - R$$

for the north star and

$$\phi = \delta + Z_o + R$$

for the south star.

It can be seen from the above expressions that the refraction term is negative in them to obtain the latitude from the north star and positive when latitude is obtained from the south star. Since the signs of the refraction terms are opposite for the north and south star latitude reductions, any constant error in the refraction model from which this term was obtained would cause an opposite effect on the latitudes computed from the two stars. If the refraction model gives a value for the refraction correction which is lower than it should be for all zenith distances, the latitude value computed for the north star will be greater than it actually is, and the latitude value from the south star will be less than it actually is. The converse case is true if the refraction model gives refraction values that are consistently higher than they should be.

In the observations which were used for this investigation, the recommended procedure had been followed in the selection of the stars. (9) This procedure calls for selecting the north and south stars at nearly equal zenith distances. When this is done, since we have seen that the refraction gives an opposite effect for the two stars, the average value of the two computed latitudes should minimize any errors due the refraction model. Since Polaris was used as the north star in these observations, south stars with a zenith distance near 50 degrees were selected.

Even when this averaging of the two latitude values is done to minimize the refraction errors, it must be pointed out that there will still be errors present in this final latitude value due to observation

and instrument conditions, but these should be smaller and in a random distribution. It should be noted that some refraction model error will still be present after the latitudes are averaged due to the fact that the stars are not at exactly the same zenith distances and possibly due to atmospheric conditions at the time of the observation.

Generally, then, it will be stated that the best refraction model would give the minimum difference between the values of latitude computed from observations of the north and south stars at any given time.

4.6 Evaluation of the Results Obtained for Latitude from the Observation Data.

It was found that the standard error of the mean value of the computed latitude was consistent for the three refraction models investigated. This is true for the latitude values computed for the north and south stars as well as the combined latitude. No significant change occurs in any of these standard error values due to the refraction model used and, in fact, very little change occurs when no refraction model at all is used. These values can be found in Table 4.

The combined latitudes for each refraction model were found to be in very close agreement for each reduction method employed, with no significant difference noted for second order results. The widest variation among these values is in the range of 0.02 seconds. Some variation was found in the computed values for the latitudes that were reduced from each of the two stars. In this light, it was found that the Coast and Geodetic Survey Model gave the least difference between the north and south star values of latitude. The Baldini Model generally gave just a slight bit more spread with the Garfinkel Model giving the widest range between these two values. This difference in

spread was found to closely equal the refraction differences. In all cases, the difference between the combined latitude value and those of the north and south stars was too great to justify using the observation or just one star to determine the astronomic latitude of a station. This is using the criteria stated in Section 4.4 for the accuracy of the observations.

The reductions were done for both the Aberdeen set of constants and the original ones suggested by Garfinkel when the Garfinkel Model was used. It was found that the Aberdeen values gave about one second less in the difference between the north star and south star latitude values for each station. This was true for all of the stations reduced. The Coast and Geodetic Survey Garfinkel program was used for the reductions also, and it was found that slightly less difference was obtained between the north and south star latitudes than when the Aberdeen program and values were used. This difference was not significant for the purposes of this investigation, having a magnitude of about 0.03 seconds of arc.

It sould be noted that the accuracy of the combined latitude would be affected to a greater degree by the refraction model as the difference in the zenith distances of the observed stars increased. At large zenith distances, the refraction model would also have more effect on the standard errors of the mean values obtained since the refraction becomes much greater as the zenith distance increases.

From the results given in Table 4, it can be stated that for second order latitudes observed using a north star and a south star with nearly equal zenith distances, any of the three refraction models investigated would give acceptable results.

TABLE 4

LATITUDE REDUCTION RESULTS
USING THE
THREE REFRACTION MODELS

Refraction Model	None	Coast and Geodetic Survey	Baldini	Garfinkel
Station:	Carr			
Temperature:	20.88°C			
Pressure:	735.2 mm			
Latitude	39° 33'	39° 32'	39° 32'	39° 32'
North Starr	"	"	"	"
Program 1	18.612	12.088	12.051	11.824
Std Error(±)	0.361	0.384	0.385	0.386
Program 2		12.191	12.154	11.927
Std Error(±)		0.385	0.386	0.386
Program 3		12.188	12.150	11.923
Std Error(±)		0.385	0.386	0.386
South Star	39° 31'			
Program 1	08.132	14.092	14.097	14.322
Std Error(±)	0.492	0.493	0.493	0.493
Program 2		14.092	14.098	14.323
Std Error(±)		0.493	0.493	0.493
Program 3		14.098	14.105	14.330
Std Error(±)		0.493	0.492	0.492
Combined	39° 32'			
Program 1	13.372	13.070	13.072	13.072
Std Error(±)	0.257	0.257	0.257	0.257
Program 2		13.141	13.125	13.125
Std Error(±)		0.257	0.257	0.257
Program 3		13.142	13.127	13.127
Std Error(±)		0.257	0.257	0.257
Difference N and S Latitudes	"	"	"	"
Program 1	130.480	2.003	2.046	2.498
Program 2		1.900	1.945	2.396
Program 3		1.910	1.955	2.406

TABLE 4 (CON'T)

LATITUDE REDUCTION RESULTS
USING THE
THREE REFRACTION MODELS
(Continued)

Refraction Model	None	Coast and Geodetic Survey	Baldini	Garfinkel
Station:	Foust			
Temperature:	16.90°C			
Pressure:	733.3 mm			
Latitude	39° 42'	39° 41'	39° 41'	39° 41'
North Star	"	"	"	"
Program 1	16.067	09.328	09.275	09.079
Std Error(±)	0.726	0.719	0.719	0.719
Program 2		09.428	09.376	09.184
Std Error(±)		0.720	0.719	0.719
Program 3		09.423	09.375	09.177
Std Error(±)		0.720	0.720	0.720
South Star	39° 40'			
Program 1	07.803	11.469	11.553	11.738
Std Error(±)	0.583	0.584	0.585	0.584
Program 2		11.472	11.556	11.742
Std Error(±)		0.584	0.585	0.584
Program 3		11.474	11.558	11.774
Std Error(±)		0.584	0.584	0.584
Combined	39° 42'			
Program 1	11.934	10.396	10.415	10.408
Std Error(±)	0.424	0.417	0.416	0.416
Program 2		10.449	10.468	10.460
Std Error(±)		0.417	0.416	0.416
Program 3		10.448	10.446	10.461
Std Error(±)		0.416	0.416	0.416
Difference N and S Latitudes	"	"	"	"
Program 1	128.264	2.141	2.278	2.659
Program 2		2.044	2.180	2.558
Program 3		2.052	2.184	2.566

TABLE 4 (CON'T)

LATITUDE REDUCATION RESULTS
USING THE
THREE REFRACTION MODELS
(Continued)

Refraction Model	None	Coast and Geodetic Survey	Baldini	Garfinkel
Station:	Haines			
Temperature:	15.88°C			
Pressure:	740.1 mm			
Latitude	39° 33'	39° 32'	39° 32'	39° 32'
North Star	"	"	"	"
Program 1	54.795	47.310	47.253	47.032
Std Error(±)	0.374	0.377	0.377	0.377
Program 2		47.396	47.337	47.116
Std Error(±)		0.377	0.378	0.378
Program 3		47.390	47.334	47.112
Std Error(±)		0.378	0.378	0.378
South Star	39° 31'			
Program 1	45.886	50.786	50.841	51.054
Std Error(±)	0.220	0.220	0.220	0.220
Program 2		50.793	50.864	51.059
Std Error(±)		0.220	0.220	0.220
Program 3		50.789	50.884	51.057
Std Error(±)		0.222	0.221	0.221
Combined	39° 32'			
Program 1	50.340	49.050	49.047	49.042
Std Error(±)	0.162	0.162	0.162	0.162
Program 2		49.092	49.092	49.086
Std Error(±)		0.161	0.162	0.161
Program 3		49.090	49.088	49.083
Std Error(±)		0.161	0.161	0.161
Difference N and S Latitudes	"	"	"	"
Program 1	128.909	3.476	3.588	4.022
Program 2		3.397	3.508	3.943
Program 3		3.398	3.610	3.945

TABLE 4 (CON'T)

LATITUDE REDUCTION RESULTS
USING THE
THREE REFRACTION MODELS
(Continued)

Refraction Model	None	Coast and Geodetic Survey	Baldini	Garfinkel
Station:	Head			
Temperature:	15.88°C			
Pressure:	738.9 mm			
Latitude	39° 32'	39° 30'	39° 30'	39° 30'
North Star	"	"	"	"
Program 1	01.838	53.382	53.301	53.110
Std Error(±)	0.304	0.294	0.294	0.295
Program 2		53.464	53.383	53.193
Std Error(±)		0.294	0.294	0.294
Program 3		53.459	53.380	53.188
Std Error(±)		0.294	0.293	0.294
South Star	39° 29'			
Program 1	50.054	55.740	55.809	55.993
Std Error(±)	0.347	0.347	0.346	0.347
Program 2		55.747	55.812	55.998
Std Error(±)		0.347	0.346	0.347
Program 3		55.745	55.812	55.996
Std Error(±)		0.346	0.346	0.346
Combined	39° 30'			
Program 1	55.944	54.561	54.554	54.551
Std Error(±)	0.275	0.266	0.265	0.265
Program 2		54.604	54.599	54.595
Std Error(±)		0.265	0.264	0.264
Program 3		54.604	54.595	54.592
Std Error(±)		0.265	0.265	0.265
Difference N and S Latitudes	"	"	"	"
Program 1	131.785	2.358	2.508	2.882
Program 2		2.283	2.429	2.804
Program 3		2.287	2.432	2.808

TABLE 4 (CON'T)

LATITUDE REDUCTION RESULTS
USING THE
THREE REFRACTION MODELS
(Continued)

Refraction Model	None	Coast and Geodetic Survey	Baldini	Garfinkel
Station:	Hoyt			
Temperature:	19.21°C			
Pressure:	738.6 mm			
Latitude	39° 40'	39° 39'	39° 39'	39° 39'
North Star	"	"	"	"
Program 1	26.719	19.949	19.953	19.740
Std Error(±)	0.420	0.456	0.455	0.455
Program 2		20.047	20.052	19.839
Std Error(±)		0.452	0.454	0.454
Program 3		20.045	20.049	19.838
Std Error(±)		0.453	0.454	0.455
South Star	39° 38'			
Program 1	18.996	22.588	22.618	22.818
Std Error(±)	0.382	0.378	0.378	0.378
Program 2		22.589	22.622	22.818
Std Error(±)		0.378	0.378	0.378
Program 3		22.589	22.622	22.824
Std Error(±)		0.378	0.379	0.378
Combined	39° 39'			
Program 1	22.857	21.266	21.283	21.279
Std Error(±)	0.339	0.347	0.348	0.348
Program 2		21.319	21.336	21.333
Std Error(±)		0.347	0.348	0.348
Program 3		21.317	21.334	21.329
Std Error(±)		0.347	0.348	0.348
Difference N and S Latitudes	"	"	"	"
Program 1	127.722	2.639	2.666	3.078
Program 2		2.542	2.570	2.994
Program 3		2.544	2.573	2.987

TABLE 4 (CON'T)

LATITUDE REDUCTION RESULTS
USING THE
THREE REFRACTION MODELS
(Continued)

Refraction Model	None	Coast and Geodetic Survey	Baldini	Garfinkel
Station:	Imler			
Temperature:	20.53°C			
Pressure:	735.7 mm			
Latitude	39° 32'	39° 31'	39° 31'	39° 31'
North Star	"	"	"	"
Program 1	43.329	36.736	36.723	36.498
Std Error(±)	0.295	0.302	0.302	0.302
Program 2		36.834	36.820	36.597
Std Error(±)		0.302	0.302	0.302
Program 3		36.829	36.817	36.594
Std Error(±)		0.302	0.302	0.302
South Star	39° 30'			
Program 1	36.789	39.541	39.576	39.785
Std Error(±)	0.187	0.187	0.187	0.187
Program 2		39.545	39.577	39.787
Std Error(±)		0.187	0.187	0.187
Program 3		39.552	39.584	39.794
Std Error(±)		0.187	0.187	0.187
Combined	39° 31'			
Program 1	40.058	38.137	38.149	38.142
Std Error(±)	0.187	0.189	0.189	0.189
Program 2		38.190	38.199	38.192
Std Error(±)		0.189	0.189	0.189
Program 3		38.190	38.201	38.194
Std Error(±)		0.189	0.189	0.190
Difference N and S Latitudes	"	"	"	"
Program 1	126.540	2.805	2.854	3.288
Program 2		2.710	2.756	3.190
Program 3		2.722	2.768	3.200

TABLE 4 (CON'T)

LATITUDE REDUCTION RESULTS
USING THE
THREE REFRACTION MODELS
(Continued)

Refraction Model	None	Coast and Geodetic Survey	Baldini	Garfinkel
Station:	Lanum			
Temperature:	12.98°C			
Pressure:	733.6 mm			
Latitude	39° 40'	39° 39'	39° 39'	39° 39'
North Star	"	"	"	"
Program 1	50.997	43.111	42.998	42.825
Std Error(±)	0.271	0.256	0.255	0.255
Program 2		43.213	43.101	42.928
Std Error(±)		0.256	0.255	0.255
Program 3		43.208	43.096	42.923
Std Error(±)		0.255	0.255	0.255
South Star	39° 38'			
Program 1	35.678	43.575	43.686	43.862
Std Error(±)	0.199	0.199	0.200	0.199
Program 2		43.578	43.690	43.862
Std Error(±)		0.199	0.199	0.199
Program 3		43.580	43.693	43.876
Std Error(±)		0.200	0.199	0.199
Combined	39° 39'			
Program 1	43.388	43.343	43.341	43.343
Std Error(±)	0.139	0.131	0.130	0.130
Program 2		43.395	43.395	43.393
Std Error(±)		0.130	0.130	0.130
Program 3		43.393	43.393	43.395
Std Error(±)		0.130	0.130	0.130
Difference N and S Latitudes	"	"	"	"
Program 1	135.219	0.464	0.688	1.036
Program 2		0.366	0.598	0.934
Program 3		0.372	0.597	0.944

TABLE 4 (CON'T)

LATITUDE REDUCTION RESULTS
USING THE
THREE REFRACTION MODELS
(Continued)

Refraction Model	None	Coast and Geodetic Survey	Baldini	Garfinkel
Station:	Meinfeller			
Temperature:	16.85°C			
Pressure:	737.3 mm			
Latitude	39° 39'	39° 37'	39° 37'	39° 37'
North Star	"	"	"	"
Program 1	07.178	59.370	59.318	59.119
Std Error(±)	0.239	0.236	0.236	0.236
Program 2		59.452	59.399	59.201
Std Error(±)		0.236	0.236	0.236
Program 3		59.449	59.394	59.196
Std Error(±)		0.236	0.236	0.236
South Star	39° 36'	39 38'	39 38'	39 38'
Program 1	55.221	00.813	00.877	01.069
Std Error(±)	0.248	0.245	0.245	0.245
Program 2		00.818	00.884	01.074
Std Error(±)		0.245	0.245	0.245
Program 3		00.818	00.884	01.074
Std Error(±)		0.245	0.245	0.245
Combined	39° 38'			
Program 1	01.199	00.092	00.097	00.094
Std Error(±)		0.159	0.159	0.159
Program 2		00.137	00.140	00.137
Std Error(±)		0.159	0.159	0.159
Program 3		00.133	00.140	00.135
Std Error(±)		0.159	0.159	0.159
Difference N and S Latitudes	"	"	"	"
Program 1	131.956	1.444	1.558	1.950
Program 2		1.366	1.485	1.873
Program 3		1.370	1.490	1.878

TABLE 4 (CON'T)

LATITUDE REDUCTION RESULTS
USING THE
THREE REFRACTION MODELS
(Continued)

Refraction Model	None	Coast and Geodetic Survey	Baldini	Garfinkel
Station:	Owens			
Temperature:	21.69°C			
Pressure:	736.4 mm			
Latitude	39° 40'	39° 39'	39° 39'	39° 39'
North Star	"	"	"	"
Program 1	42.374	36.151	36.216	35.982
Std Error(±)	0.348	0.360	0.361	0.361
Program 2		36.254	36.315	36.084
Std Error(±)		0.360	0.361	0.361
Program 3		36.252	36.315	36.082
Std Error(±)		0.360	0.361	0.361
South Star	39° 38'			
Program 1	33.001	39.283	39.222	39.455
Std Error(±)	0.526	0.514	0.513	0.514
Program 2		39.295	39.235	39.469
Std Error(±)		0.509	0.509	0.509
Program 3		39.275	39.218	39.448
Std Error(±)		0.519	0.519	0.519
Combined	39° 39'			
Program 1	37.685	37.716	37.718	37.716
Std Error(±)	0.301	0.282	0.281	0.281
Program 2		37.774	37.773	37.774
Std Error(±)		0.282	0.281	0.281
Program 3		37.762	37.766	37.766
Std Error(±)		0.285	0.284	0.284
Difference N and S Latitudes	"	"	"	"
Program 1	129.373	3.133	3.006	3.473
Program 2		3.042	2.920	3.385
Program 3		3.023	2.903	3.366

TABLE 4 (CON'T)

LATITUDE REDUCTION RESULTS
USING THE
THREE REFRACTION MODELS
(Continued)

Refraction Model	None	Coast and Geodetic Survey	Baldini	Garfinkel
Station:	Persinger			
Temperature:	19.72°C			
Pressure:	741.2 mm			
Latitude	39° 33'	39° 32'	39° 32'	39° 32'
North Star	"	"	"	"
Program 1	20.578	13.098	13.039	12.820
Std Error(±)	0.264	0.275	0.276	0.276
Program 2		13.199	13.137	12.921
Std Error(±)		0.276	0.277	0.277
Program 3		13.197	13.135	12.917
Std Error(±)		0.275	0.276	0.277
South Star	39° 31'			
Program 1	09.767	14.775	14.838	15.046
Std Error(±)	0.567	0.561	0.561	0.561
Program 2		14.782	14.785	15.055
Std Error(±)		0.559	0.559	0.559
Program 3		14.776	14.842	15.051
Std Error(±)		0.562	0.562	0.562
Combined	39° 32'			
Program 1	15.171	13.937	13.937	13.935
Std Error(±)	0.314	0.307	0.307	0.307
Program 2		13.989	13.992	13.985
Std Error(±)		0.307	0.306	0.307
Program 3		13.987	13.989	13.985
Std Error(±)		0.307	0.307	0.307
Difference N and S Latitudes	"	"	"	"
Program 1	130.811	1.777	1.779	2.227
Program 2		1.503	1.708	2.133
Program 3		1.580	1.706	2.134

TABLE 4 (CON'T)

LATITUDE REDUCTION RESULTS
USING THE
THREE REFRACTION MODELS
(Continued)

Refraction Model	None	Coast and Geodetic Survey	Baldini	Garfinkel
Station:	Tarbill			
Temperature:	24.01°C			
Pressure:	737.9 mm			
Latitude	39° 33'	39° 32'	39° 32'	39° 32'
North Star	"	"	"	"
Program 1	08.683	02.767	02.788	02.532
Std Error(±)	0.395	0.385	0.385	0.385
Program 2		02.870	02.887	02.633
Std Error(±)		0.385	0.385	0.385
Program 3		02.685	02.884	02.630
Std Error(±)		0.385	0.385	0.385
South Star	39° 31'			
Program 1	02.341	04.563	04.570	04.806
Std Error(±)	0.505	0.510	0.510	0.510
Program 2		04.570	04.573	04.813
Std Error(±)		0.511	0.511	0.511
Program 3		04.571	04.578	04.817
Std Error(±)		0.509	0.509	0.509
Combined	39° 32'			
Program 1	05.510	03.665	03.677	03.670
Std Error(±)	0.165	0.154	0.154	0.154
Program 2		03.720	03.730	03.723
Std Error(±)		0.155	0.154	0.154
Program 3		03.720	03.730	03.722
Std Error(±)		0.155	0.154	0.154
Difference N and S Latitudes	"	"	"	"
Program 1	126.343	1.796	1.782	2.274
Program 2		1.700	1.686	2.180
Program 3		1.706	1.694	2.187

TABLE 4 (CON'T)

LATITUDE REDUCATION RESULTS
USING THE
THREE REFRACTION MODELS
(Continued)

Refraction Model	None	Coast and Geodetic Survey	Baldini	Garfinkel
Station:	Taylor			
Temperature:	21.45°C			
Pressure:	732.0 mm			
Latitude	39° 41'	39° 40'	39° 40'	39° 40'
North Star	"	"	"	"
Program 1	38.837	32.646	32.639	32.407
Std Error(±)	0.680	0.689	0.690	0.690
Program 2		32.730	32.722	32.490
Std Error(±)		0.690	0.691	0.691
Program 3		32.728	32.718	32.490
Std Error(±)		0.690	0.691	0.691
South Star	39° 39'			
Program 1	31.123	35.329	35.355	35.576
Std Error(±)	0.294	0.291	0.291	0.291
Program 2		35.388	35.362	35.585
Std Error(±)		0.291	0.291	0.291
Program 3		35.334	35.360	35.583
Std Error(±)		0.292	0.292	0.292
Combined	39° 40'			
Program 1	34.977	33.987	33.995	33.992
Std Error(±)	0.289	0.296	0.296	0.296
Program 2		34.031	34.038	34.036
Std Error(±)		0.296	0.297	0.297
Program 3		34.031	34.038	34.035
Std Error(±)		0.295	0.296	0.296
Difference N and S Latitudes	"	"	"	"
Program 1	127.715	2.683	2.716	3.168
Program 2		2.607	2.641	3.095
Program 3		2.606	2.642	3.093

TABLE 4 (CON'T)

LATITUDE REDUCTION RESULTS
USING THE
THREE REFRACTION MODELS
(Continued)

Refraction Model	None	Coast and Geodetic Survey	Baldini	Garfinkel
Station:	Williams			
Temperature:	16.75°C			
Pressure:	730.4 mm			
Latitude	39° 33'	39° 32'	39° 32'	39° 32'
North Star	"	"	"	"
Program 1	33.210	25.497	25.377	25.179
Std Error(±)	0.367	0.370	0.370	0.370
Program 2		25.524	25.404	25.207
Std Error(±)		0.370	0.370	0.370
Program 3		25.519	25.401	25.203
Std Error(±)		0.370	0.371	0.371
South Star	39° 31'			
Program 1	26.538	28.509	28.619	28.800
Std Error(±)	0.607	0.599	0.599	0.599
Program 2		28.513	28.626	28.806
Std Error(±)		0.597	0.597	0.597
Program 3		28.515	28.626	28.805
Std Error(±)		0.601	0.601	0.601
Combined	39° 32'			
Program 1	29.874	27.002	26.999	26.987
Std Error(±)	0.343	0.336	0.336	0.335
Program 2		27.019	27.016	27.006
Std Error(±)		0.334	0.334	0.334
Program 3		27.016	27.013	27.004
Std Error(±)		0.335	0.336	0.336
Difference N and S Latitudes	"	"	"	"
Program 1	126.672	3.012	3.242	3.621
Program 2		2.988	3.222	3.600
Program 3		2.996	3.226	3.602

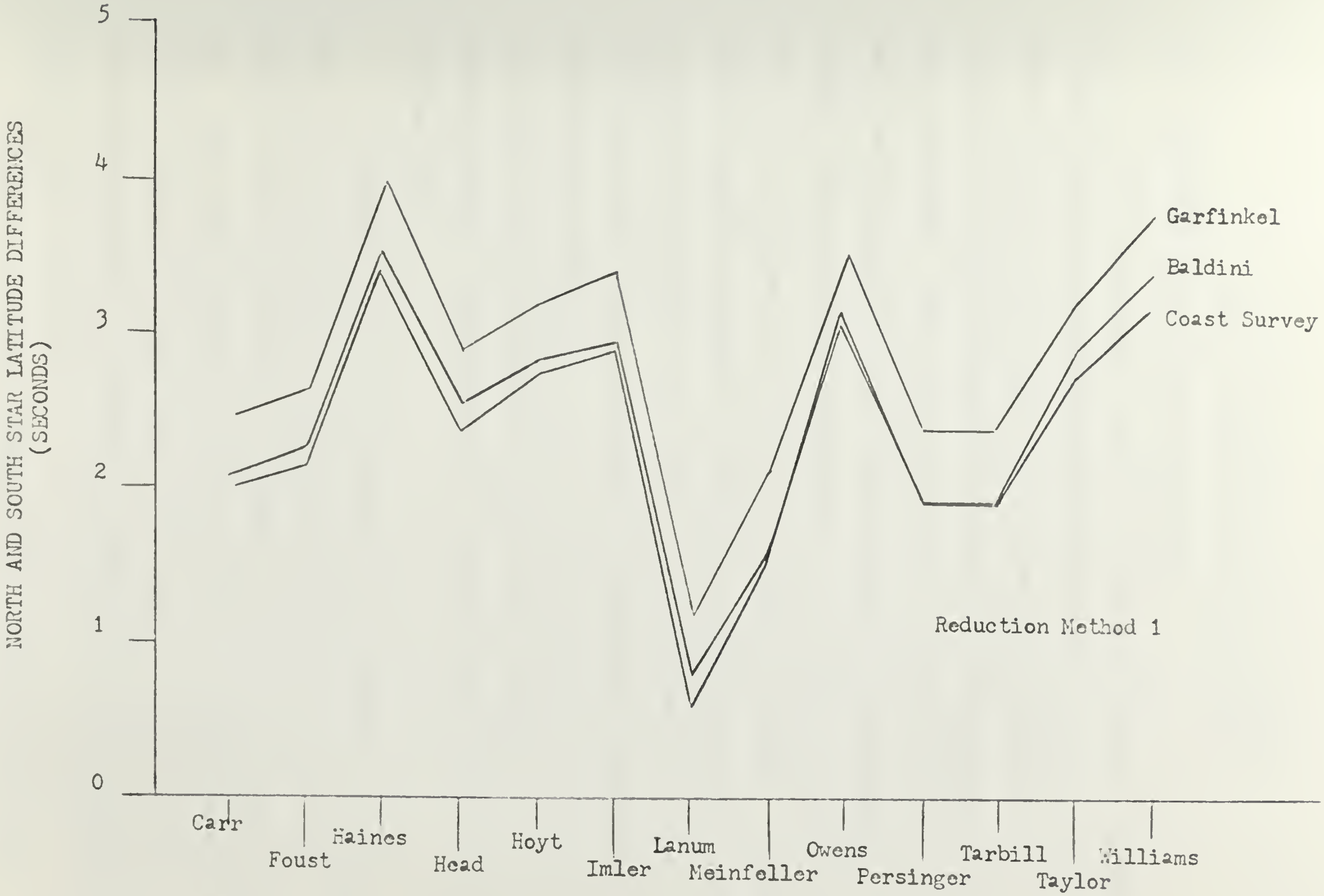


Figure 6. North and South Star Latitude Differences at Observation Stations

CHAPTER 5

A SUGGESTION FOR AN ADJUSTMENT PROCEDURE USING A FORM OF THE GARFINKEL MODEL

5.1 Background for the Adjustment Method

Since the atmosphere and its refraction effects can be varied by the conditions which prevail concerning weather, season or turbulance at any given time, there will be some factors which can not be accounted for by the assumptions for which the refraction model has been constructed. These effects will not be exceptionally large, but their presence can effect the results obtained for the refraction at a station. There also may be some errors in the refraction model itself that can be minimized by the use of an adjustment procedure. It might then be possible to construct an adjustment program which would perform an adjustment on certain of the refraction coefficients in order to give refraction results which will reflect the conditions at the station where the observations were taken. If many stations were treated in this manner, then the coefficients used could be examined for any general trends exhibited. This could lead to revisions and a more accurate refraction model.

To truly reflect the differences in the refraction coefficients and to take into account the effect of any observation errors, the General Method of Adjustments[10] should be used for this type of project. This method considers both the variation of assumed parameters and the effect of the observations. This system was used in an investigation conducted to determine the feasibility of this suggested type of station adjustment.

In the investigation conducted, two variations of the same

basic form were employed. The first form used the condition that the values of latitude computed from the observation of the north and south stars should be the same. The second form is more general and assumes that the first general equation derived from spherical trigonometry and the astronomical triangle, given in Section 4.1, should be satisfied for the observed star after correction for the refraction. These two methods will be discussed individually in the sections to follow.

5.2 The Form of the Garfinkel Refraction Model Selected

The form of the Garfinkel Refraction model suggested by ACIC (4) was chosen for this adjustment procedure. This form was discussed in Section 3.4. The refraction is then obtained from the relation that

$$R = W\sqrt{To}\ (N_1 \tan\theta + N_2 \tan^3\theta + N_3 \tan^5\theta)$$

where

$$W = P/{To}^2$$

$$\tan\theta = \frac{To}{\gamma} \operatorname{Tan} Z_o$$

and the value of γ is chosen as 8.7137. All of these quantities have been previously defined. For the purposes of an adjustment, it was considered that the pressure, temperature and gamma factor were constant. The gamma factor used here is not the one suggested in the ACIC article but is that of the Aberdeen and Coast and Geodetic Survey Garfinkel programs which was found to give less difference in the computed latitudes of the north and south stars. Since $\tan Z_o$ is the same as $\cot h_o$, the observed altitude was substituted into the adjustment equations for the observed zenith distance. The assumed parameters become N_1, N_2 and N_3 from this equation. These factors were assumed from the refraction results given by the Garfinkel Model for zenith distances in the 40 to 60 degree range, with an approximate mean

value taken for each factor. These assumed values are given in Appendix II along with an explanation of the adjustment procedures.

5.3 Adjustment Method Using a Mathematical Structure Dealing with the Latitudes Computed from the North and South Stars

In this method the mathematical structure is

$$\phi_N - \phi_S = 0$$

where the subscripts denote from which observed star the latitude was computed. The declination is considered as a fixed quantity in this adjustment. The observed quantities are considered to be the hour angle and altitude of both the north and south stars. This results in a B matrix with dimensions of 4 by 8 which consists of the partial derivatives of the mathematical structure with respect to each of the observed quantities. The weight of all of the observations has been considered to be equal. The parameters are considered to be the three factors from the refraction equation and the assumed latitude used for south star computations. This gives an A matrix of dimensions 4 by 8 which consists of the partial derivatives of the mathematical structure with respect to the parameters. No weights were assigned the parameters for this adjustment. The basic formulas used to obtain the corrections to the parameters are given as

$$AX + BV + W = 0$$

$$X = -(A'M^{-1}A)^{-1}A'MW$$

$$M = BB'$$

where the X matrix gives the corrections to the assumed parameters.[10] The W matrix is computed for each observation and each element is the number which would have to be added to the equation, computed from the observed values and assumed parameters, to fulfill the condition given by the mathematical structure. The components for these matrices and

and some additional information on this adjustment method are given in Appendix II.

5.4 Adjustment Method with a More General Mathematical Structure.

The second method used the same computed quantities but a different mathematical structure that gives a condition which each observed star must fulfill. This method has the same parameters and fixed values as the previous method but has a B matrix with the dimensions 2 by 16, since there are only two observed quantities for each observation. These are the hour angle and the altitude of each observed star. The same final adjustment matrix equations are used for this form as for the previous one. This is a more general form and if it were found workable would give more room for variation in the stars observed as two south stars could be substituted for the north and south star of the previous method. The same thing could be done using two north stars also. This method could be expanded to use more data than that provided by just two stars because of its general form.

The investigation for this model used only two stars due to the lack of any other available data. The matrix components used and other information on the adjustment are given in Appendix II.

5.5 Results of the Adjustment Method Investigation

The results obtained from the investigation of these two suggested adjustment methods are not in any way conclusive in a positive sense. It was found that the observed data were too strongly correlated to give any useable results. This was caused by a very unstable M matrix which in turn caused the inverse M matrix to be non-symmetrical and therefore caused the adjustment results to be unuseable. This correlation may be due to the fact that all of the observed altitudes

were within two degrees of each other.

Therefore, nothing can be said in the way of constructive conclusions from the adjustment results except that they do not work for the data available for this project. A suggested positive test for these methods would be to observe stars, both north and south of the zenith, at different zenith distances and hour angles and then use the adjustment procedures. The reduction procedures for these observations would not be the same as those used in this project and new equations would have to be written with the mathematical structure given in Section 5.3, as the forms given in Appendix II are for the case of Polaris and a circummeridian star. A difference in zenith distances of from 20 to 30 degrees is recommended. These zenith distances should be observed up to about 70 degrees for the study.

If this type of station adjustment for the refraction coefficients was found feasible, then a sequence of observations, say on two stars, could be taken before and after the latitude observations at the station. It would be best if all observations were taken at a single station over a long period of time to determine any consistent differences in the refraction model. However, data taken at different stations could be used in the project for comparison purposes.

If it were proved to be feasible, this type of station adjustment might eventually lead to a situation where the latitude of a station could be determined, for second order accuracy, with the observation of only one star. The station adjustment could also be employed as a tool for determining seasonable variations and their effects on the refraction model. Therefore, the idea does have some merit and should be investigated rather throughly before being completely discarded.

CHAPTER 6

SUMMARY AND CONCLUSIONS

The basic objective of this project was to determine if the refraction model used in second order astronomic latitude determinations had any effect on the final latitude results. The method of observing both a north and south star at nearly the same zenith distance was used. It can be stated, based upon the results of this project, that the final combined astronomic latitude is not significantly affected by the refraction model used. This is in reference to the three refraction models used in this investigation. It may also be noted that the standard error of the mean for the computed latitudes is not affected to any significant degree by the refraction model used in reductions.

It was found that with the three refraction models used, that it would not be feasible to observe just one star, north or south, with an expectation of obtaining results of acceptable second order accuracy. This is based on the consideration that a standard error of ± 0.6 seconds of arc would be the acceptable criteria. The spread given between the latitude values computed for the north and south stars was greater than this value in each station computed and for each refraction model.

If the difference or spread obtained between the north and south star latitudes can be used as a criteria, it can be seen that the Garfinkel Model gives a consistently larger difference, at the zenith distances observed (about 50 degrees), than either the Coast and Geodetic Survey Model or the Baldini Model. These latter two models give results very close to each other in all the stations computed.

In a comparison of the refraction models at various temperatures

and pressures, which ranged to extremes of each quantity. It was found that the Coast and Geodetic Survey and the Baldini Models are consistently closer to each other in the refraction for a specified zenith distance, except for zenith distances over 80 degrees, than the results obtained from the Garfinkel Model. Since the Baldini Model was only designed for zenith distances up to 75 degrees, this seems to be a good comparison. It must be noted, however, that up to a zenith distance of 50 degrees, the difference in the refractions given by the three models is less than 0.3 seconds of arc. The Garfinkel Model is the only one of the three that were investigated which is designed to give refraction for zenith distances of 90 degrees or larger.

The Baldini Model was investigated for the effects of relative humidity and the wave length of the observed light. It was found that the humidity did not have a very significant effect on the refraction output of the model until a zenith distance of about 70 degrees was reached. The difference in wave lengths of the received light causes a greater variation, but its effect is diminished when it is considered that nearly all the observable stars give off light in the yellow wave length region of the spectrum. Consequently, the Baldini model was used with a relative humidity of 60 percent and a wave length of yellow light equivalent to that used for the basic theory of the Coast and Geodetic Survey Model. The results for the Baldini Model would have been a bit more accurate if the relative humidity had been taken at the observing stations and the wave length of the observed light known. It is considered that, for the purposes of this project, these factors would not have had a very great influence.

It was attempted to ascertain if an adjustment for the con-

ditions at the observing station could be made to improve the refraction model. Sufficient data were not available to determine if this adjustment was feasible. It is suggested that this area could be one of future research but that data should be taken at a variety of zenith distances to determine if the adjustment method would actually improve the refraction results.

Finally, it may be said that the three refraction models investigated in this project are adequate for second order latitude determination employing the present two star observation methods; however, it must be noted that none of these refraction models gives the ultimate solution to the refraction problem and, therefore, this area still remains open to future investigation.

APPENDIX

I

REFRACTION PROGRAM INFORMATION

In all of the refraction subprograms used in this investigation and listed in this Appendix, the input and output values are in the same units. For the calling parameters the observed zenith distance is in radians, the temperature is in degrees centigrade, the pressure is in inches of mercury and the relative humidity is given as a decimal fraction. All the subroutines give the output for refraction in seconds of arc. The requirements of each specific subroutine will now be listed.

The Coast and Geodetic Survey Subroutine requires that the values of refraction from Table V of U. S. C.& G S Special Publication 237 [3] be read into the main program. Since these are listed for every ten minutes of arc this consists of reading in 541 values. At the same time a corresponding quantity in degrees is generated by the program. The Subroutine is called with the calling parameters in the following order: Relative Humidity, Temperature, Pressure, Zenith Distance. Literals statements are included in the subprogram to determine the factors for temperature and pressure conditions due to deviation from the standard conditions assumed for the computation of the tables. These values are from Tables VI and VII of the publication mentioned above.

The Baldini Subroutine uses a literals statement to include a table from which the water vapor pressure at the observing station may be computed. This is Table I of the Baldini article.[1] The value "Slite" in the program is the wave length of light used in the computations. This subprogram is called in the following order: Relative Humidity, Temperature, Pressure, Zenith Distance.

All of the Garfinkel Subroutines listed in this Appendix have the same calling parameters. These are called in the following order: Zenith Distance, Pressure, Temperature. In all of the programs the gamma factor and other constants used are included through the use of a literals statement.

The program for a comparison of the refraction models is fairly straightforward. It reads in the values required for the Coast and Geodetic Survey Subroutine and then proceeds to call each of the subroutines given above to determine the refraction for various temperature and pressure combinations. Since these combinations range to extremes, the Coast and Geodetic Survey Subroutine had to be extended to include temperature and pressure correction factors for the entire range. This was done by simply expanding the literals statement for these factors to cover more range of the respective tables they were taken from. This expanded form is not shown here.

The last item in this Appendix is a comparison of the various versions of the Garfinkel Subprogram. The Aberdeen and Coast and Geodetic Survey comparisons were made for the newer constants and a gamma factor of 8.7137. The other two comparisons used Garfinkel's original constants and a gamma factor of 8.1578.

PROGRAM STATEMENTS AND SCATRAN SUBROUTINE
FOR THE
U. S. COAST AND GEODETIC SURVEY REFRACTION MODEL

```
        DIMENSION (RM(541),ZA(541))-

  START READ IMPUT, USREF, ((RM(K),K=0,1,K.LE540))-

F USREF (12F5.1)-

        DO THROUGH (TABLE),I=0,1,I.L.540-

        DO THROUGH (TABLE),J=0,1,J.L.6-

        IJ = I + J-

        ZA(IJ) = (I/6.0) + (J * 0.1666666667)-

  TABLE CONTINUE-

        ZA(540) = 90.0-

        SUBROUTINE (RCGS) = REFCGS. (HUM,TEMP,PRESS,ZRS)-

        DIMENSION (CENT(46),CTM(46),BAR(30),CBM(30))-

        UNIVERSAL (RM,ZA)-

        ZDR = ZRS * 57.2957797-

        LITERALS (CENT,5.0,5.6,6.1,6.7,7.2,7.8,8.3,8.9,9.4,10.0,
        10.6,11.1,11.7,12.2,12.8,13.3,13.9,14.4,15.0,15.6,16.1,
        16.7,17.2,17.8,18.3,18.9,19.4,20.0,20.6,21.1,21.7,22.2
        22.8,23.3,23.9,24.4,25.0,25.6,26.1,26.7,27.2,27.8,28.3,
        28.9,29.4,30.0)-

        LITERALS (CTM,1.018,1.016,1.014,1.012,1.010,1.008,1.006,
        1.004,1.002,1.000,0.998,0.996,0.994,0.992,0.990,0.988,
        0.986,0.985,0.983,0.981,0.979,0.977,0.975,0.973,0.972,
        0.970,0.968,0.966,0.964,0.962,0.961,0.959,0.957,0.955,
        0.953,0.952,0.950,0.948,0.946,0.945,0.943,0.941,0.939,
        0.938,0.936,0.934)-

        LITERALS (BAR,28.1,28.2,28.3,28.4,28.5,28.6,28.7,28.8,
        28.9,29.0,29.1,29.2,29.3,29.4,29.5,29.6,29.7,29.8,29.9,
        30.0,30.1,30.2,30.3,30.4,30.5,30.6,30.7,30.8,30.9,31.0)-

        LITERALS (CBM,0.939,0.942,0.946,0.949,0.953,0.956,0.959,
        0.963,0.966,0.970,0.973,0.976,0.979,0.983,0.986,0.989,
        0.992,0.996,0.999,1.003,1.007,1.010,1.013,1.016,1.020,
        1.023,1.026,1.029,1.033,1.036)-
```

```
        CT = 0-

        DO THROUGH (GEDT), IG=0,1,IG.L.45-

        JG = IG + 1-

        PROVIDED (TEMP.GE.CENT(IG).AND.TEMP.LE.CENT(JG)),
        CT = CTM(IG) + (CTM(JG) - CTM(IG)) * (TEMP - CENT(IG))/
        ((CENT(JG) - CENT(IG))-

        PROVIDED (CT.G.0),TRANSFER (G01)-

GEDT    CONTINUE-

G01     CB = 0-

        DO THROUGH (GEDP), IB=0,1,IB.L.29-

        JB = IB + 1-

        PROVIDED (PRESS.GE.BAR(IB).AND.PRESS.LE.BAR(JB)),
        CB = CBM(IB) + (CBM(JB) - CBM(IB)) + (PRESS -
        BAR(IB))/(BAR(JB) - BAR(IB))-

        PROVIDED (CB.G.0.), TRANSFER (G02)-

GEDP    CONTINUE-

G02     RT = 0-

        DO THROUGH (GETZ), I=0,1,I.L.540-

        J = I + 1-

        PROVIDED (ZDR.GE.ZA(I).AND.ZDR.LE.ZA(J)), RT = RM(I) +
        (RM(J) - RM(I)) * (ZDR - ZA(I))/(ZA(J) - ZA(I))-

        PROVIDED (RT.G.0.), TRANSFER (FINAL)-

GEDZ    CONTINUE-

FINAL   RCGS = RT * CT * CB-

        NORMAL EXIT-

        END SUBPROGRAM-

        END PROGRAM (START)-
```

SCATRAN SUBROUTINE
FOR THE
BALDINI REFRACTION MODEL

```
        SUBROUTINE (RB) = REFBAL. (RH,TDC,ATPRS,ZST)-

        DIMENSION (HEAT(10),HG(10))-

        PATM = ATPRS * 25.39998-

        LITERALS (HEAT,-5.0,0.0,5.0,10.0,15.0,20.0,25.0,30.0,
        35.0,40.0)-

        LITERALS ( HG,3.02,4.58,6.54,9.21,12.79,17.55,23.78,
        31.86,42.23,55.40)-

        HGMM = 0 -

        DO THROUGH (VAPOR), IF = 0,1,IF.L.9-

        JF = IF + 1-

        PROVIDED (TDC.GE.HEAT(IF).AND.TDC.LE.HEAT(JF)), HGMM=
        HG(IF) +(HG(JF) - HG(IF)) * (TDC - HEAT(IF))/5.0-

        PROVIDED (HGMM.G.0), TRANSFER (CONT)-

VAPOR   CONTINUE -

CONT    E = RH * HGMM-

        SLITE = 0.578-

        AO = 0.99827-

        A1 = -0.00130-
        A2 = 0.000006

        FNG = 2876.04 = 16.288/(SLITE.P.2) + 0.136?(SLITE.P.4)-

        FNG = FNG * 1.0.X.-7-

        FNO = (FNG/(1.0 + 0.00367 * TDC)) * PTAM/760.0 -
        (5.5.X.-8) * E/(1.0 + 0.00367 * TDC)-

        FNO = FNO * 206264.81-

        TAN = SIN.(ZST)/COS.(ZST)-

        RB = FNO * AO * TAN + FNO * A1 * (TAN.P.3) + FNO * A2 *
        (TAN.P.5)-
```

NORMAL EXIT-

END SUBPROGRAM-

SCATRAN SUBROUTINE
FOR THE
GARFINKEL REFRACTION MODEL
ABERDEEN VERSION

```
       SUBROUTINE (R) = REFGAR. (ZOB,PP,TM)-
       CALL SUBROUTINE ( ) = FPSET. (3,HALT)-
HALT   CONTINUE-
       LITERALS (FN,4.256,FK,4472.8,BE,0.044385,GAMA,8.7137)-
       UPPER COMMON (IND)-
       BOOLEAN (IND)-
       DIMENSION (B(5))-
       PR = PP/760.0 * 25.3998-
       T = TM/273.16 + 1.0-
       W = PR/T/T-
       TRANSFER (GR15,GR12,GR12) PROVIDED (ZOB - 0.7853981634)-
GR15   TANTA = SIN.(ZOB)/COS.(ZOB)) * SQRT.(T)/GAMA-
       THETA = ATAN.(TANTA)-
       TRANSFER (GR120)-
GR12   TANTA = GAMA * (COS.(ZOB)/SIN.(ZOB))/SQRT.(T)-
       THETA = ATAN.(1.0/TANTA)-
       PROVIDED (TANTA.E.0), THETA = ZOB-
GR120  PS = 0.0-
       S = 0.0-
       SUM = 0.0-
       DO THROUGH (GR33), I=1,1,I.LE.5-
       DO THROUGH (GR32), J=1,1,J.LE.I-
       FI = I -1-
       FJ = J -1-
       FM = FN * (FI - FJ + 1.0) - FI-
```

```
        FJJ = FJ + 1.0-
        CALL SUBROUTINE ( ) = GAMMA. (FIJJ,FIJJ)-
        PROVIDED (IND), TRANSFER (GR200)-
        FIJJ = FI - FJ + 1.0-
        CALL SUBROUTINE ( ) = GAMMA. (FIJJ,FIJJ)-
        PROVIDED (IND), TRANSFER (GR200)-
        FMS = FM + 1.0-
        CALL SUBROUTINE ( ) = GAMMA. (FM,FMG)-
        PROVIDED (IND), TRANSFER (GR200)-
        CALL SUBROUTINE ( ) = GAMMA. (FMS,FMS)-
        PROVIDED (IND), TRANSFER (GR200)-
        D = FMG * FMS-
GR100   TRANSFER (GR13,GR31,GR13) PROVIDED (FM - S)-
GR13    D = (FM + S)/(FM - S) * D-
        KK = 2.0 * S + 1.0-
        CH = COS.(THETA/2.0).P.KK-
        SH = SIN.(THETA/2.0).P.KK-
        TA = SH/CH-
        TRANSFER (GR132, GR131, GR131) PROVIDED (SH - 0.70710678)-
GR131   CN = CH/SH-
        TA = 1.0/CN-
GR132   P1 = FMG * FMG/D * TA-
        TRANSFER (GR14, GR30, GR14) PROVIDED (FM - S - 1.0)-
GR14    D = (FM + S + 1.0)/(FM - S - 1.0) * D-
        KK + S * 2.0 + 3.0-
        ST = SIN.(THETE/2.0).P.KK-
        CT = COS.(THETE/2.0).P.KK-
```

```
        TN = ST/CT-
        TRANSFER (GR142, GR141, GR141) PROVIDED (ST - 0.7.710678)-
GR141   CN = CT/ST-
        TN = 1.0/CN-
GR142   P1 + P1 + FMG * FMG/D * TN-
GR143   PS = PS + P1-
        TRANSFER (GR16, GR31, GR16) PROVIDED (P1)-
GR16    S = S + 2.0-
        TRANSFER (GR17, GR24, GR17) PROVIDED (S - 2.0)-
GR17    TRANSFER (GR24, GR24, GR30) PROVIDED (S - FM)-
GR24    P = P1-
        TRANSFER (GR100)-
GR30    PS = PS + P1-
GR31    FMNIF = FM + FI-
        CALL SUBROUTINE ( ) = GAMMA. (FMNIF,FMNIF)-
        FMM = (-1.0).P.(J-1) * FMNIF/(FJJ * FIJJ * FMG)-
        SUM = SUM + FMM * PS-
        PS = 0.0-
GR32    S = 0.0-
        B(I) = FK * BE.P.(I-1) * SUM-
GR33    SUM = 0.0-
        R = SQRT.(T) * W * (B(1( + W * (B(2) + W * (B(3) + W *
        (B(4) + W * (B(5)))))-
        NORMAL EXIT-
GR200   CALL SUBROUTINE ( ) = ENDJOB. ( )-
        END SUBPROGRAM-
        SUBROUTINE ( ) = GAMMA. (Z,GZ)-
```

```
          LITERALS (C1,0.001917526918,C2,0.0008417508418,
          C3,0.0005952381,C4,0.0007936507936508,C5,0.0027777777777,
          C6,0.08333333333,C7,0.9189358533)-

          UPPER COMMON (IND)-

          BOOLEAN (IND)-

          X = Z-

          CV = 1.0-

   GA9    TRANSFER (GA10, GA11, GA11) PROVIDED (X -10.0)-

   GA10   CV = CV * X-

          X = X +1.0-

          TRANSFER (GA9)-

   GA11   TRANSFER (GA12, GA80, GA12) PROVIDED (CV)-

   GA12   TRANSFER (GA10, GA13, GA13) PROVIDED (.ABS.CV - 1.0)-

   GA13   ZZ = 1.0/X/X-

          FX = (((((C1 * ZZ) + C2) * ZZ - C3) * ZZ + C4) * ZZ - C5)
          * ZZ + C6-

          FX = FX/X-

          G = (X - 0.5) * LN.(X) + C7 - X + FX-

          TRANSFER (GA14, GA90, GA90) PROVIDED (G - 700.0)-

   GA14   GZ = EXPE. (G)/CV-

          NORMAL EXIT-

   GA80   WRITE OUTPUT, F81, (Z)-

F  F81    (34H GAMMA OF ZERO OR NEGATIVE INTEGER, E14.8)-

          IND(=).0.77777777777-

          NORMAL EXIT-

   GA90   WRITE OUTPUT, F91, (Z)-

F  F91    (44H GAMMA FUNCTION WILL EXCEED MACHINE CAPACITY, E24.8)-

          IND(=).0.77777777777-

          NORMAL EXIT-
```

END SUBPROGRAM-

SCATRAN SUBROUTINE
FOR THE
GARFINKEL REFRACTION MODEL
U. S. COAST AND GEODETIC SURVEY VERSION

```
SUBROUTINE (R) = REFGAR. (ZZ,PR,TT)-

FLOATING (N1,N2,N3,N4)-

LITERALS (N1,1050.61030,N2,706.11502,N3,262.06086,
N4,142.67293,GAMMA,8.7137)-

T = TT/273.16 = 1.0-

P = PR/760.0 * 25.39998-

W = P/(T * T)-

TANB = (SQRT.(T).GAMMA) * (SIN.(ZZ)/COS.(ZZ))-

B = ATAN.(TANB)-

BH = B/2.0-

TANBH = SIN.(BH)/COS.(BH)-

FT = W * (SQRT.(T))-

R = FT * ((N1 * TANBH) + (N2 * (TANBH.P.3)) +
(N3 * (TANBH.P.5)) + (N4 * (TANBH.P.7)))-

NORMAL EXIT-

END SUBPROGRAM-
```

SCATRAN SUBROUTINE
FOR THE
GARFINKEL REFRACTION MODEL
ACIC FORM

```
SUBROUTINE (REF) = REFGAR. (ZEN,PRS,TM)-

DIMENSION (FN1(61),FN2(50),FN3(15))-

LITERALS (FN1,0.0,8.7,17.3,25.9,34.6,43.3,52.0,60.7,69.5,
78.3,87.1,96.0,104.9,113.8,122.9,131.9,141.0,150.2,159.6,
168.9,178.3,187.8,197.4,207.1,216.9,226.9,237.0,247.1,
257.4,267.9,278.5,289.2,300.1,311.2,322.4,333.9,345.9,
357.3,369.4,381.7,394.2,407.0,420.0,433.3,446.9,460.8,
475.0,489.6,504.5,519.7,535.4,551.5,567.9,584.9,602.4,
620.3,638.8,657.8,677.4,697.7,718.6)-

LITERALS (FN2,0.0,0.0,0.0,0.1,0.1,0.1,0.1,0.1,0.2,0.2,
0.2,0.3,0.3,0.4,0.4,0.5,0.6,0.7,0.7,0.8,0.9,1.0,1.1,1.2,
1.4,1.5,1.7,1.9,2.1,2.3,2.5,2.7,3.0,3.3,3.6,3.9,4.3,4.7,
5.1,5.5,5.9,6.5,7.0,7.6,8.3,9.0,9.7,10.5,11.4,12.4)-

LITERALS (FN3,0.0,0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.2,0.2,
0.2,0.2,0.3,0.3,0.4)-

TS = 273.0/(TM + 273.0)-

T = SQRT.(TS)-

GAMMA = 8.1578-

P = (PRS/760.0) * 25.39998-

W = P * TS * TS-

COTZ = COS.(ZEN)/SIN.(ZEN)-

GAMO = GAMMA * T-

TAN = 1.0/(GAMO * COTZ)-

THETA = ATAN.(TAN)-

ANG = THETA * 57.2957795-

DO THROUGH (ONE), I=0,1,I.L.60-

J = I + 1-

PROVIDED (ANG.L.I.OR.ANG.G.J.), TRANSFER (ONE)-

B1 = FN1(I) + ((FN1(J) -FN1(I)) * (ANG - I))-
```

```
        PROVIDED (I.LE.10), TRANSFER (SET1)-
        IK + I - 11-
        JK = J - 11-
        B2 = FN2(IK) + ((FN2(JK) - FN2(IK)) * (ANG - I))-
        PROVIDED (I.LE.45), TRANSFER (SET2)-
        IM = I - 46-
        JM = J - 46-
        B3 + FN3(IM) + ((FN3(JM) - FN3(IM)) * (ANG - I))-
        TRANSFER (OUT)-
SET1    B2 = 0.0-
SET2    B3 = 0.0-
        TRANSFER (OUT)-
ONE     CONTINUE-
OUT     REF = (W*B1)/T + (W * W * B2)/T + (W * W * W * B3)/T-
        NORMAL EXIT-
        END SUBPROGRAM-
```

SCATRAN PROGRAM
FOR THE
COMPARISON OF THE REFRACTION
GIVEN BY THE THREE MODELS
FOR VARIOUS TEMPERATURE
AND PRESSURE COMBINATIONS

```
        DIMENSION (RM(541),ZA(541))-
  START READ IMPUT, USREF, ((RM(K), K=0,1,K.LE.540))-
F USREF (12F5.1)-
        DO THROUGH (TABLE), I=0,1,I.L.540-
        DO THROUGH (TABLE), J=0,1.J.L.6-
        IJ = I + J-
        ZA(IJ) = (I/6.0) + (J * 0.166666667)-
  TABLE CONTINUE-
        ZA(540) = 90.0-
        RHUM = 0.60-
        DIMENSION (CENT(3),PRSMM(4))-
        LITERALS (CENT,-5.0,10.0,35.0)-
        LITERALS (PRSMM,711.0,737.0,760.0,787.0)-
        DO THROUGH (END), I=0,1,I.L.3-
        CTEM = CENT(I)-
        DO THROUGH (END), J=0,1,J.L.4-
        WRITE OUTPUT, F1-
        PRSIN = PRSMM (J)/25.39998-
        DO THROUGH (ONE), K=0,1,K.L.19-
        FACT = K-
        ZD = 5.0 * FACT-
        ZRAD = ZD/57.2957795-
```

```
          CALL SUBROUTINE (COAST) = REFCGS. (RHUM,CTEM,PRSIN,ZRAD)-

          CALL SUBROUTINE (BAID) = REFBAL. (RHUM,CTEM,PRSIN,ZRAD)-

          CALL SUBROUTINE (GARF) = REFGAR. (ZRAD,PRSIN,CTEM)-

          WRITE OUTPUT, F2, (ZD,COAST,BALD,GARF)-

   ONE    CONTINUE-

          WRITE OUTPUT, F3, (CENT(I),PRSMM(J))-

   END    CONTINUE-

F  F1     (1H1,Q$ COMPARISON OF REFRACTION FROM 0 TO 90 DEGREES AT
          FIVE DEGREE INTERVALS$//Q$ ZENITH DISTANCE $,5X,Q$ COAST
          AND GEODETIC SURVEY MODEL$,5X,Q$ BALDINI MODEL$,5X,Q$
          GARFINKEL MODEL$)-

F  F2     (//6X,F4.1,20X,F10.6,19X,F10.6,10X,F10.6)-

F  F3     (//Q$ ABOVE VALUES ARE IN SECONDS OF ARC$//Q$ THIS DATA
          WAS COMPUTED FOR A WAVE LENGTH OF 0.578 MICRONS$//Q$
          RELATIVE HUMIDITY EQUALS 60 PER CENT$//Q$ CENTIGRADE
          TEMPERATURE EQUALS$,F7.1//Q$ ATMOSPHERIC PRESSURE EQUALS$,
          F7.1,1X,Q$ millimeters$)-

          CALL SUBROUTINE ( ) = ENDJOB. ( )-

          U. S. COAST AND GEODETIC SURVEY SUBROUTINE
          (MODIFIED FOR A LARGER RANGE OF TEMPERATURE AND PRESSURE)

          BALDINI SUBROUTINE

          GARFINKEL SUBROUTINE

          END PROGRAM (START)-
```

COMPARISON OF THE GARFINKEL MODEL SUBROUTINES

Zenith Distance (degrees)	Aberdeen Version	Coast and Geodetic Survey Version	Aberdeen Version Original Constants	ACIC Form
0	0.000	0.000	0.000	0.000
5	5.090	5.088	5.123	5.157
10	10.257	10.254	10.325	10.367
15	15.587	15.582	15.690	15.702
20	21.171	21.164	21.311	21.288
25	27.120	27.112	27.300	27.266
30	33.574	33.564	33.796	33.784
35	40.711	40.699	40.980	40.979
40	48.775	48.760	49.097	49.091
45	58.107	58.091	58.492	58.463
50	69.216	69.198	69.674	69.685
55	82.887	82.867	83.435	83.422
60	100.414	100.393	101.079	101.059
65	124.100	124.801	124.922	124.922
70	158.465	158.457	159.418	159.554
75	213.740	213.770	215.169	215.100
80	318.633	318.825	320.830	320.933
85	589.651	590.955	594.659	594.666

Temperature = 10° Centegrade
Pressure = 760.0 mm of Hg

Refraction in Seconds of Arc

APPENDIX

II

ADJUSTMENT OF THE REFRACTION MODEL
FOR STATION CONDITIONS

This adjustment procedure makes use of the General Method of Adjustments to obtain a solution for the variation of the assumed parameters.[10] The general matrix form may be represented in matrix notation as

$$\begin{bmatrix} BP^{-1}B' & A \\ A' & 0 \end{bmatrix} \begin{bmatrix} K_L \\ X \end{bmatrix} = \begin{bmatrix} W \\ 0 \end{bmatrix}$$

or may be given in a matrix equation as

$$BV + AX + W = 0.$$

In these forma all letters represent matrices (with the exception of the subscript). The A, B and W matrices are defined as

$$A = \frac{\partial F}{\partial X_o}$$

$$B = \frac{\partial F}{\partial L_b}$$

$$W = F(L_b, X_o)$$

where L_b represents the observed quantities, L_a represents the adjusted values of the observed quantities, X_o represents the assumed parameters and X_a represents the adjusted values for the parameters. Other matrices used in the solution of the equations are the V matrix which gives the residuals for the observation, the X matrix which gives the corrections to the assumed parameters and the P matrix which is the weight matrix. The prime symbols on the matrix indicate that it is the transpose of the matrix whose letter is used and the minus one notation indicates that this matrix is the inverse of the matrix whose letter is used. Defining a new matrix as

$$M = BP^{-1}B'$$

tho final form of the equation for the X matrix is given as

$$X = -(A'M^{-1}A)^{-1} A'M^{-1}W \quad .$$

For an adjustment to take place the condition must be satisfied that

$$n + u > r > u$$

where n is the number of observations of the observed quantities, u is the number of unknown parameters and r is the number of condition equations. When these conditions are fulfilled then the adjustment can take place.[10]

The above factors and principles are then an outline of the method which was used to attempt to adjust the refraction coefficients for the conditions at the stations.

Two mathematical structures were employed in the adjustment attempt with two seperate attempts being used to make the adjustment. In the first instance the observed quantities were the observed altitude and hour angle of each star. This gave four observed quantities per condition. In the second instance only two observed quantities were required, these being the altitude and azimuth for the single star inquestion. In both cases the parameters were the three refraction model coefficients and tho assumed latitude. The declination, temperature, pressure and time between direct and reverse sightings of the instrument were considered not to vary for the purposes of the adjustment.

The equation used for the refraction model was based on the ACIC form of the Garfinkel Model which is given here as

$$R = \sqrt{T_o}\; W(N_1 K \cot h_o + N_2 K^3 \cot^3 h_o + N_3 K^5 \cot^5 h_o)$$

where

$$W = P/T_0^2$$

$$K = T_o/\gamma$$

with γ assigned the Aberdeen value of 8.7137. In this form P and T_0 are the relative pressures and temperatures. These quantities are obtained from the relation

$$T_o = \frac{T_o + 273.16}{273.16}$$

$$P = P/760.0$$

where the observed temperature is in degrees centigrade and the observed pressure is in millimeters of mercury. The cotangent function appears in the equation since

$$\tan\theta = \frac{T_0}{\gamma} \tan Z_0 = \frac{T_0}{\gamma} \cot h_0$$

where Z_0 is the observed zenith distance and h_0 is the observed altitude.

The relation may also be established that

$$h = h_0 + R$$

where h is the observed altitude after correction for the refraction and R is the refraction correction from the refraction model. These relationships will be used in the normal equation coefficients which are derived for each mathematical structure.

The first mathematical structure investigated was based on the fact that, if all things were perfect, the latitude obtained from an observation of a south circummeridian star would be the same as that obtained from the observation of Polaris. This is fitting the more general statement that the latitude obtained from the north star should be the same as the latitude obtained from the south star to our available data. The basic condition is then that the difference of these two values be equal to zero. This mathematical structure is given as

$$\phi_N - \phi_S = 0$$

with the subscripts denoting the north and south stars. The reduction methods used by Thorson[9] were used to obtain these values in terms of the observed values and parameters. Using Polaris, the latitude from the north star observation becomes

$$\phi_N = h - P \cos t + B$$

where

$$P = (90^\circ - \delta)$$

$$B = \tfrac{1}{2}P^2 \tan h \sin^2 t$$

with t representing the hour angle, δ representing the declination and h being the corrected altitude of Polaris. The latitude from the circummeridian star may be represented as

$$\phi_S = \delta - h' + 90^\circ$$

where

$$h' = h + Am$$

$$m = m_1 + m_2 = 0.0001364(\Delta T^s)^2 + \sin^2\tfrac{1}{2}t$$

$$A = \frac{\cos\delta\cos\phi}{\cos h}$$

with h representing the corrected altitude, t given as the hour angle of the south star, δ representing the declination and ΔT^s given as the time in seconds between the direct and reverse pointings of the instrument. Now denoting the north star quantities with an asterisk and the south star quantities with no superscript the mathematical structure may be written as

$$h_o^* - R^* - P\cos t^* + \tfrac{1}{2}P^2 \tan(h_o^* - R^*)\sin^2 t^*$$
$$-90^\circ - \delta + h - R + \frac{\cos\delta\cos\phi}{\cos(h_o - R)}\ 0.0001364(\Delta T^s)^2 + 2\sin^2\tfrac{1}{2}t = 0.$$

Since eight observations are taken of the north and south stars individually, this mathematical structure gives eight conditions to be fulfilled. There are four assumed parameters and a total of thirty-two observed quantities. This gives the values

$$n = 32$$

$$r = 8$$

$$u = 4$$

which when used to ascertain if an adjustment will take place gives the relation

$$(32 + 4) > 8 > 4.$$

Therefore, an adjustment can exist for this amount of data used in the above mathematical structure.

The A and B matrices are the ones whose form we are concerned with here. All observations are taken to be of the same weight which causes the P matrix to be a unit matrix which may be dropped from the equations. It should be noted here that the values of the altitude and hour angle are correlated and that possible weights should be assigned to take account of this fact. The A matrix takes the form

$$\begin{bmatrix} \frac{\partial F}{\partial N_1} & \frac{\partial F}{\partial N_2} & \frac{\partial F}{\partial N_3} & \frac{\partial F}{\partial \phi} \end{bmatrix}$$

and the B matrix is of the form

$$\begin{bmatrix} \frac{\partial F}{\partial h_o^*} & \frac{\partial F}{\partial t_o^*} & \frac{\partial F}{\partial h_o} & \frac{\partial F}{\partial t_o} \end{bmatrix}.$$

For simplification in the writing of the formulas for these partial derivatives some factors will be designated for use in the formulas. These factors are

$$FR = N_1 \frac{\sqrt{T_o}}{\gamma} \frac{1}{\sin^2 h_o} + 3N_2 \left(\frac{\sqrt{T_o}}{\gamma}\right)^3 \frac{\cot^2{}_o}{\sin^2 h_o} + 5N_3 \left(\frac{\sqrt{T_o}}{\gamma}\right)^5 \frac{\cot^4 h_o}{\sin^2 h_o}$$

$$FW = W\sqrt{T_o}$$

$$FN = \frac{\sqrt{T_o}}{\gamma} \cot h_o$$

$$FA = \frac{\cos\delta \cos\phi \sin(h_o - R)}{\cos^2(h_o - R)}$$

$$FPOL = \frac{\frac{1}{2}P^2 \sin^2 t^*}{\cos^2(h^* - R^*)}$$

$$FCIR = \frac{\sin(h_o - R)}{\cos(h_o - R)}$$

which will be used along with the factors A and m defined above.

The partial derivitaves for the A and B matrices may now be written

$$\frac{\partial F}{\partial h_o^*} = 1 + FW(FR^*) + FPOL\left[1 + FW(FR^*)\right]$$

$$\frac{\partial F}{\partial t^*} = P \sin t^* + P^2 \tan(H^* - R^*) \sin t^* \cos t^*$$

$$\frac{\partial F}{\partial h_o} = 1 + FW(FR) + \frac{Am}{\cos(h_o - R)}\left[1 + FW(FR)\right]$$

$$\frac{\partial F}{\partial t} = \frac{\cos\delta \cos\phi}{\cos(h_o - R)}\left(2 \sin\tfrac{1}{2}t \cos\tfrac{1}{2}t\right)$$

$$\frac{\partial F}{\partial N_1} = -FW(FN^*) - FW(FN^*)FPOL - FW(FN)(Am)FCIR$$

$$\frac{\partial F}{\partial N_2} = -FW(FN^*)^3 - FW(FN^*)^3\, FPOL - FW(FN)^3 - FW(FN)^3(Am)FCIR$$

$$\frac{\partial F}{\partial N_3} = -FW(FN^*)^5 - FW(FN^*)^5\, FPOL - FW(FN)^5 - FW(FN)^5(Am)FCIR$$

$$\frac{\partial F}{\partial \phi} = -Am \tan\phi$$

where the values with the askerisk once again represent values computed from the north star and those same values without a superscript represent these quantities computed from the south star. These derivitives are rather involved and a simpler system would be more desirable if it could be found.

A simpler and more general method of expressing the mathematical structure is to use the first equation for the astronomical triangle

given in Section 4.1. This gives the form

$$\sin\phi \sin\delta + \cos\phi \cos\delta \cos t - \sin h = 0$$

where the quantities used in the formula are the same as those which were described above. In this method, each star observation must fulfill the condition given by the mathematical structure.

For this method there are thirty two observed quantities, six teen conditions to be fulfilled and four unknowns. The data used for this method was the same as in the case of the first mathematical structure given above. The assumed parameters remain the same in this adjustment and therefore there is no change in the general form of the A matrix. The B matrix is reduced to two terms since only the observation of a single star is considered for each condition. This lets the B matrix assume the form

$$\begin{bmatrix} \dfrac{\partial F}{\partial t} & \dfrac{\partial F}{\partial h_o} \end{bmatrix}$$

where t is now the hour angle of the observed star and h_o the observed altitude with no distinction being made for the north and south stars. For this method the values

$$n = 32$$

$$r = 16$$

$$u = 4$$

are obtained which give the relation

$$(32 + 4) > 16 > 4.$$

This relation shows that an adjustment can exist for this mathematical structure with the available data.

The partial dorivitaves for this method are easier to work with than those which were taken from the previous mathematical structure. Using the constant factors FW, FH and FR from above the partial

derivatives become

$$\frac{\partial F}{\partial t} = -\cos\phi \cos\delta \sin t$$

$$\frac{\partial F}{\partial h_o} = \cos(h_o - R)\left[1 + FW(FR)\right]$$

$$\frac{\partial F}{\partial N_1} = -\cos(h_o - R)(FW)(FN)$$

$$\frac{\partial F}{\partial N_2} = -\cos(h_o - R)(FW)(FN)^3$$

$$\frac{\partial F}{\partial N_3} = -\cos(h_o - R)(FW)(FN)^5$$

$$\frac{\partial F}{\partial \phi} = \cos\phi \sin\delta - \sin\phi \cos\delta \cos t$$

with these values being computed for each observed star.

In both of these adjustment methods given above the W matrix can be computed from the mathematical structure. Then with the A, B and W matrices the matrix equations can be used for each adjustment and a solution should be obtained if the data are not too strongly correlated.

The values for the refraction model coefficients were selected as

$$N_1 = 522.5$$
$$N_2 = 86.9$$
$$N_3 = 8.2$$

from an examination of the Garfinkel model output for zenith distances from 40 to 60 degrees. The numerical values given above are an average of all the coefficient values obtained by this process.

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